[edit] Bell's thought experiment

Bell considered a setup in which two observers, Alice and Bob, perform independent measurements on a system **S** prepared in some fixed state. Each observer has a detector with which to make measurements. On each trial, Alice and Bob can independently choose between various detector settings. Alice can choose a detector setting a to obtain a measurement $A(a)$ and Bob can choose a detector setting b to measure $B(b)$. After repeated trials Alice and Bob collect statistics on their measurements and correlate the results.

There are two main assumptions in Bell's analysis: (1) Each measurement reveals an objective physical property of the system and (2) A measurement taken by one observer has no effect on the measurement taken by the other.

In the language of probability theory, repeated measurements of system properties can be regarded as repeated sampling of random variables. One might expect measurements by Alice and Bob to be somehow correlated with each other: the random variables are assumed not to be independent, but linked in some way. Nonetheless, there is a limit to the amount of correlation one might expect to see. This is what the Bell inequality expresses.

A version of the Bell inequality appropriate for this example is given by John Clauser, Michael Horne, Abner Shimony and R. A. Holt, and is called the CHSH form:

$$(1)\quad \mathbf{C}[A(a),B(b)]+\mathbf{C}[A(a),B(b')]+\mathbf{C}[A(a'),B(b)]-\mathbf{C}[A(a'),B(b')]\leq 2,$$

where **C** denotes correlation.

[edit] Importance of the theorem

This theorem has even been called "the most profound in science" (Stapp, 1975). Bell's seminal 1964 paper was entitled "On the Einstein Podolsky Rosen paradox". The Einstein Podolsky Rosen paradox (EPR paradox) assumes local realism, the intuitive notion that particle attributes have definite values independent of the act of observation and that physical effects have a finite propagation speed. Bell showed that local realism leads to a requirement for certain types of phenomena that are not present in quantum mechanics. This requirement is called **Bell's inequality**.

After EPR (Einstein-Podolsky-Rosen), quantum mechanics was left in an unsatisfactory position: either it was incomplete, in the sense that it failed to account for some elements of physical reality, or it violated the principle of finite propagation speed of physical effects. In a modified version of the EPR thought experiment, two observers, now commonly referred to as *Alice* and *Bob*, perform independent measurements of spin on a pair of electrons, prepared at a source in a special state called a *spin singlet state*. It was a conclusion of EPR that once Alice measured spin in one direction (e.g. on the x axis), Bob's measurement in that direction was determined with certainty, whereas immediately before Alice's measurement, Bob's outcome was only statistically determined. Thus, either the spin in each direction is not an element of physical reality, or the effects travel from Alice to Bob instantly.

In QM, predictions were formulated in terms of probabilities â " for example, the probability that an electron might be detected in a particular region of space, or the probability that it would have spin up or down. The idea persisted, however, that the electron in fact has a **definite** position and spin, and that QM's weakness was its inability to predict those values precisely. The possibility remained that some yet unknown, but more powerful theory, such as a *hidden variables theory*, might be able to predict those quantities exactly, while at the same time also being in complete agreement with the probabilistic answers given by QM. If a *hidden variables theory* were correct, the hidden variables were not described by QM, and thus QM would be an incomplete theory.

The desire for a *local realist theory* was based on two ideas: first, that objects have a definite state that determines the values of all other measurable properties, such as position and momentum; and second, that (as a result of special relativity) effects of local actions, such as measurements, cannot travel faster than the speed of light. In the formalization of local realism used by Bell, the predictions of theory result from the application of classical probability theory to an underlying parameter space. By a simple (but clever) argument based on classical probability, he then showed that correlations between measurements are bounded in a way that is violated by QM.

Bell's theorem seemed to put an end to local realist hopes for QM. Per Bell's theorem, either quantum mechanics or local realism is wrong. Experiments were needed to determine which is correct, but it took many years and many improvements in technology to perform them.

Bell test experiments to date overwhelmingly show that the inequalities of Bell's theorem are violated. This provides empirical evidence against local realism. They are also taken as positive evidence in favor of QM. The principle of special relativity is saved by the no-communication theorem, which proves that the observers cannot use the inequality violations to communicate information to each other faster than the speed of light.

John Bell's papers examined both John von Neumann's 1932 proof of the incompatibility of hidden variables with QM and Albert Einstein and his colleagues' seminal 1935 paper on the subject.

Correlation of observables *X*, *Y* is defined as

$$\mathrm{C}(X, Y) = \mathrm{E}(XY).$$

This is non-normalized form of the correlation coefficient considered in statistics (see Quantum correlation).

In order to formulate Bell's theorem, we formalize local realism as follows:

1. There is a probability space Î› and the observed outcomes by both Alice and Bob result by random sampling of the parameter $\lambda \in \Lambda$.
2. The values observed by Alice or Bob are functions of the local detector settings and the hidden parameter only. Thus
   - Value observed by Alice with detector setting *a* is *A*(*a*,Î»)
   - Value observed by Bob with detector setting *b* is *B*(*b*,Î»)

[edit] Bell's theorem: Bell inequalities are violated by some quantum predictions

To finish Bell's theorem we will show that quantum mechanics makes a prediction that violates a "Bell inequality" in the setup considered in the EPR thought experiment. In order to do this, we first need to show how to compute correlations of quantum mechanical observables.

In the usual quantum mechanical formalism, observables *X*, *Y* are represented as self-adjoint operators on a Hilbert space. To compute the correlation, assume that *X*, *Y* are represented by matrices in a finite dimensional space and that *X*, *Y* commute; this special case suffices for our purposes below. We then use the von Neumann measurement postulate: a series of measurements of an observable *X* on a series of identical systems in state Ï† produces a distribution of real values. By the assumption that observables are finite matrices, this distribution is discrete. The probability of observing Î» is non-zero if and only if Î» is an eigenvalue of the matrix *X* and moreover the probability is

$$\|\mathrm{E}_X(\lambda)\phi\|^2$$

(where E_X (Î») is the projector corresponding to the eigenvalue Î». The system state immediately after the measurement is

$$\|\mathrm{E}_X(\lambda)\phi\|^{-1}\mathrm{E}_X(\lambda)\phi.$$

From this, we can show that the correlation of commuting observables *X*, *Y* in a pure state Ïˆ is

$$\langle XY \rangle = \langle XY\psi \mid \psi \rangle.$$

We apply this fact in the context of the EPR paradox. The measurements performed by Alice and Bob are spin measurements for an electron. Alice can choose between two detector settings labelled *a* and *a*â ²; these settings correspond to measurement of spin along the *z* or the *x* axis. Bob can choose between two detector settings labelled *b* and *b*â ²; these correspond to measurement of spin along the *z*â ² or *x*â ² axis, where the *x*â ² â " *z*â ² coordinate system is rotated 45Â° relative to the *x* â " *z* coordinate system. The spin observables are represented by the 2 Ã— 2 self-adjoint matrices:

$$S_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$S_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

These are the Pauli spin matrices normalized so that the corresponding eigenvalues are +1, â^'1. As is customary, we denote the eigenvectors of S_x by

$$|+x\rangle, \quad |-x\rangle.$$

Let Ït be the spin singlet state for a pair of electrons discussed in the EPR paradox. This is a specially constructed state described by the following vector in the tensor product

$$|\phi\rangle = \frac{1}{\sqrt{2}}\Big(|+x\rangle \otimes |-x\rangle - |-x\rangle \otimes |+x\rangle \Big).$$

Now let us apply the CHSH formalism to the measurements that can be performed by Alice and Bob.

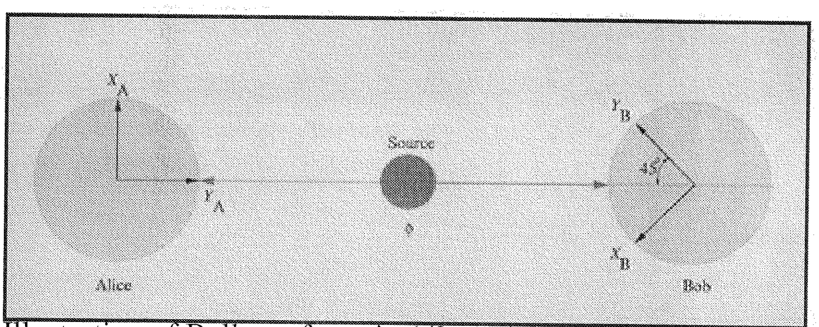

Illustration of Bell test for spin 1/2 particles. Source produces spin singlet pair, one particle sent to Alice another to Bob. Each performs one of the two spin measurements.

$$A(a) = S_z \otimes I$$

$$A(a') = S_x \otimes I$$

$$B(b) = -\frac{1}{\sqrt{2}}\, I \otimes (S_z + S_x)$$

$$B(b') = \frac{1}{\sqrt{2}}\, I \otimes (S_z - S_x).$$

The operators *B*(*b*'), *B*(*b*) correspond to Bob's spin measurements along *x*â ² and *z*â ². Note that the *A* operators commute with the *B* operators, so we can apply our calculation for the correlation. In this case, we can show that the CHSH inequality fails. In fact, a straightforward calculation shows that

$$\langle A(a)B(b)\rangle = \langle A(a')B(b)\rangle = \langle A(a')B(b')\rangle = \frac{}{\sqrt{2}},$$

and

$$\langle A(a)B(b')\rangle = -\frac{1}{\sqrt{2}}.$$

so that

$$\langle A(a)B(b)\rangle + \langle A(a')B(b')\rangle + \langle A(a')B(b)\rangle - \langle A(a)B(b')\rangle = \frac{4}{\sqrt{2}} = 2\sqrt{2} > 2.$$

[edit] Notable quotes

Heinz Pagels, in *The Cosmic Code*, writes:

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Some recent popularizers of Bell's work when confronted with [Bell's inequality] have gone on to claim that telepathy is verified or the mystical notion that all parts of the universe are instantaneously interconnected is vindicated. Others assert that this implies communication faster than the speed of light. That is rubbish; the quantum theory and Bell's inequality imply nothing of this kind. Individuals who make such claims have substituted a wish-fulfilling fantasy for understanding. If we closely examine Bell's experiment we will see a bit of sleight of hand by the God that plays dice which rules out actual nonlocal influences. Just as we think we have captured a really weird beast--like acausal influences--it slips out of our grasp. The slippery property of quantum reality is again manifested.