EVALUATION OF $d^{(j)}_{mm'}(\beta)$ MATRICES

$j = 1/2$ WE HAVE ALREADY SEEN: $d^{1/2} = \begin{pmatrix} \cos(\beta/2) & -\sin(\beta/2) \\ \sin(\beta/2) & \cos(\beta/2) \end{pmatrix}$

HOW ABOUT $j = 1$

QUITE OBVIOUSLY, WE MUST HAVE

$\dim = \#(\text{POSSIBLE } m \text{ VALUES})$

$= 2j + 1 = 3$

FROM DEFINITIONS

$\boxed{J_+ = J_x + iJ_y}$ $\boxed{J_- = J_x - iJ_y}$

$$\Longrightarrow J_y = \frac{J_+ - J_-}{2i}$$

THE MATRIX ELEMENTS WE KNOW:

$$\langle j',m'|J_+|j,m\rangle = \sqrt{(j-m)(j+m+1)}\,\hbar \times \delta_{jj'}\,\delta_{m',m+1}$$

$$\langle j'm'|J_-|j,m\rangle = \sqrt{(j+m)(j-m+1)}\,\hbar\,\delta_{jj'}\,\delta_{m'm-1}$$

AND SO $J_y = \dfrac{J_+ - J_-}{2i}$ IS SUM OF 2 MATRICES

$$\langle 1,m'|J_-|1,m\rangle = \sqrt{(1+m)(2-m)}\,\hbar\,\delta\delta$$

$$= \sqrt{2+m-m^2}\,\hbar\,\delta_{m',m-1}$$

$$\begin{array}{c|ccc|c} m & 1 & 0 & -1 & m' \\ \hline & 0 & 0 & 0 & 1 \\ & \sqrt{2} & 0 & 0 & 0 \\ & 0 & \sqrt{2} & 0 & -1 \end{array}$$

$\sqrt{2-m-m^2}$

$$\langle 1,m'|J_+|1,m\rangle = \sqrt{(1-m)(2+m)}\,\hbar\,\delta_{m'm+1}$$

$$\begin{array}{c|ccc|c} m & 1 & 0 & -1 & m' \\ \hline & 0 & \sqrt{2} & 0 & 1 \\ & 0 & 0 & \sqrt{2} & 0 \\ & 0 & 0 & 0 & -1 \end{array}$$

$$J_y^1 = \frac{\hbar}{2i}\begin{pmatrix} 0 & +\sqrt{2} & 0 \\ -\sqrt{2} & 0 & +\sqrt{2} \\ 0 & -\sqrt{2} & 0 \end{pmatrix}$$

$$= \frac{\hbar}{2}\begin{pmatrix} 0 & -i\sqrt{2} & 0 \\ i\sqrt{2} & 0 & -i\sqrt{2} \\ 0 & i\sqrt{2} & 0 \end{pmatrix}$$

$$d^1 = \exp\left(-iJ_y\beta/\hbar\right)$$

AGAIN LETS CALCULATE THE TAYLOR EXPANSION

FOR $\frac{J_y}{\hbar} = \frac{i}{2}\begin{pmatrix} 0 & -\sqrt{2} & 0 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix}$

$$\frac{J_y}{\hbar} \cdot \frac{J_y}{\hbar} = \frac{i^2}{4}\begin{pmatrix} -2 & 0 & 2 \\ 0 & -4 & 0 \\ 2 & 0 & -2 \end{pmatrix} = \frac{-1}{2}\begin{pmatrix} -1 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

$$\left(\frac{J_y}{\hbar}\right)^3 = \frac{-i}{4}\begin{pmatrix} 0 & 2\sqrt{2} & 0 \\ -2\sqrt{2} & 0 & 2\sqrt{2} \\ 0 & -2\sqrt{2} & 0 \end{pmatrix} = \frac{i}{2}\begin{pmatrix} 0 & -\sqrt{2} & 0 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix}$$

$$\Longrightarrow \left(\frac{J_y}{\hbar}\right)^3 = \left(\frac{J_y}{\hbar}\right)$$

THUS ODD POWERS GO

$\frac{J_y}{\hbar}$ AND CONTRIBUTION

$-i\left(\frac{J_y}{\hbar}\right)\sin(\beta)$

EVEN TERMS GO WITH

COS EXCEPT THE K=0

$\rightarrow 1+\left(\frac{J_y}{\hbar}\right)^2\cos(\beta)-\left(\frac{J_y}{\hbar}\right)^2$

$$\Rightarrow e^{-\frac{iJ_y\beta}{\hbar}} = 1-\left(\frac{J_y}{\hbar}\right)^2(1-\cos\beta) - i\left(\frac{J_y}{\hbar}\right)\sin\beta$$

$$= \begin{pmatrix} \frac{1}{2}(1+c\beta) & -\frac{1}{\sqrt{2}}s\beta & \frac{1}{2}(1-c\beta \\ \frac{1}{\sqrt{2}}s\beta & c\beta & -\frac{1}{\sqrt{2}}s\beta \\ \frac{1}{2}(1-c\beta) & \frac{1}{\sqrt{2}}s\beta & \frac{1}{2}(1+c\beta \end{pmatrix}$$