# SIMPLE HARMONIC OSCILLATOR

- QUICK MEMORY REFRESHMENT
  IN ORDER TO DEMONSTRATE THE NATURE OF UNCERTAINTY PRINCIPLE
- MORE THOROUGH TREATMENT CAN BE FOUND FROM LECTURE MATERIAL OF QUANTUM MECHANICS II

- HAMILTONIAN

$$H = \frac{1}{2m}P^2 + \frac{1}{2}m\omega^2 X_0^2$$

$\omega = \sqrt{k/m}$ = ANGULAR FREQUENCY OF A CLASSICAL OSCILLATOR WHOSE SPRING CONSTANT IS $k$

LET US DEFINE

- ANNIHILATION OPERATOR

$$a = \sqrt{\frac{m\omega}{2\hbar}} \left( x + \frac{i p}{m\omega} \right)$$

- CREATION OPERATOR

$$a^{\dagger} = \sqrt{\frac{m\omega}{2\hbar}} \left( x - \frac{i p}{m\omega} \right)$$

$a, a^{\dagger}$ NON-HERMITIAN

- USING CANONICAL COMMUTATION RELATIONS $[x_i, p_j] = i\hbar \delta_{ij}$

$[x_i, x_j] = [p_i, p_j] = 0$

WE GET

$$\boxed{[a, a^{\dagger}] = \mathbb{1}} \Rightarrow \boxed{[a^{\dagger}, a] = -\mathbb{1}}$$

LET US DEFINE

$$\boxed{\underline{\text{NUMBER OPERATOR}} \quad N = a^\dagger a}$$ HERMITIAN

WHICH CAN BE SHOWN TO BE

$$N = \frac{H}{\hbar\omega} - \frac{1}{2}$$

$$\Rightarrow \boxed{H = \hbar\omega(N + 1/2)}$$

OBVIOUSLY $[H, N] = 0$

LET US DENOTE AN ENERGY EIGENKET $|E_n\rangle$ BY ITS EIGENVALUE $n$:

$$\boxed{N|n\rangle = n|n\rangle} \Rightarrow \boxed{H|n\rangle = (n + 1/2)\hbar\omega|n\rangle}$$

$$\Rightarrow \boxed{E_n = (n + 1/2)\hbar\omega}$$

DIRECT CALCULATION

$$\boxed{[N, a] = -a} \quad \boxed{[N, a^\dagger] = a^\dagger}$$

AND SO

$$\boxed{\begin{aligned} N a^\dagger |n\rangle &= (n+1) a^\dagger |n\rangle \\ N a |n\rangle &= (n-1) a |n\rangle \end{aligned}}$$

TO BE COMPARED WITH

$$N |n-1\rangle = (n-1) |n-1\rangle$$

APPARENTLY

$$a|n\rangle = c|n-1\rangle$$

A COMPLEX NUMBER

REMEMBER: $|n-1\rangle$ AND $c|n-1\rangle$ REPRESENT THE SAME STATE

NORMALIZATION

$$\langle n|n\rangle = 1$$

$$\langle n|a^\dagger a|n\rangle$$

$$= \langle n-1|c^* c|n-1\rangle$$

$$= |c|^2$$

$$\langle n|\underbrace{a^\dagger a}_{=N}|n\rangle = \langle n|n|n\rangle = n$$

$$\Longrightarrow \boxed{|c|^2 = n}$$

LET'S CHOOSE $c \in \mathbb{R}e$ AND $c > 0$

(4)

AND SO

$$\boxed{a|n\rangle = \sqrt{n}\,|n-1\rangle}$$

$$\boxed{a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle}$$

AND SO $a$ AND $a^\dagger$ TRULY DESERVE THEIR NAMES AS THEY DESTROY AND CREATE A QUANTUM UNIT OF ENERGY ($\hbar\omega$)

NEGATIVE ENERGY SOLUTIONS FORBIDDEN

$\rightarrow$ $n$ MUST BE NON-NEGATIVE INTEGER

AND GROUND STATE IS

$$\boxed{E_0 = \frac{1}{2}\hbar\omega}$$

THE MATRIX ELEMENTS READ

$$\langle n'|a|n\rangle = \sqrt{n}\,\delta_{n',n-1}$$

$$\langle n'|a^{\dagger}|n\rangle = \sqrt{n+1}\,\delta_{n',n+1}$$

AND THUS ⊛

$$\langle n'|x|n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n}\,\delta_{n',n-1} + \sqrt{n+1}\,\delta_{n',n+1}\right)$$

$$\langle n'|p|n\rangle = \sqrt{\frac{m\hbar\omega}{2}}\left(-\sqrt{n}\,\delta_{n',n-1} + \sqrt{n+1}\,\delta_{n',n+1}\right)$$

BY INVERTING THE DEFINITIONS OF $a$ AND $a^{\dagger}$ WE GET

$$\boxed{x = \sqrt{\frac{\hbar}{2m\omega}}\left(a + a^{\dagger}\right)} \qquad \boxed{p = i\sqrt{\frac{m\hbar\omega}{2}}\left(a^{\dagger} - a\right)}$$

⊛ INSERT THESE

- FINALLY WE GET TO THE POINT...

- THE EXPECTATION VALUES ARE EVALUATED <u>WITH RESPECT TO SOME STATE</u>

$$\langle x \rangle = \langle \alpha | x | \alpha \rangle$$

$$\langle p \rangle = \langle \alpha | p | \alpha \rangle$$

OBVIOUSLY $\langle x \rangle = \langle p \rangle \equiv 0$ FOR ALL STATES OF SIMPLE HARMONIC OSCILLATOR

- THUS, IN ORDER TO CALCULATE THE DISPERSIONS

$$\langle (\Delta x)^2 \rangle = \langle x^2 \rangle - \langle x \rangle^2$$

$$\langle (\Delta p)^2 \rangle = \langle p^2 \rangle - \langle p \rangle^2$$

WE NEED TO EVALUATE $\langle x^2 \rangle$ AND $\langle p^2 \rangle$ ONLY

$$X^2 = XX = \frac{\hbar}{2m\omega}(a+a^\dagger)(a+a^\dagger)$$

$$= C_x\left(a^2 + aa^\dagger + a^\dagger a + a^{\dagger 2}\right)$$

$$X^2|n\rangle = C_x\left(a^2|n\rangle + aa^\dagger|n\rangle + a^\dagger a|n\rangle + a^{\dagger 2}|n\rangle\right)$$

$$= C_x\left(\sqrt{n}\sqrt{n-1}\,|n-2\rangle + \sqrt{n+1}^{\,2}|n\rangle\right.$$

$$\left. + \sqrt{n}\sqrt{n}\,|n\rangle + \sqrt{n+1}\sqrt{n+2}\,|n+2\rangle\right)$$

AND SO

$\langle n|X^2|n\rangle$ CONTRIBUTIONS COME FROM 2ND AND 3RD TERM

$$\underline{\langle n|X^2|n\rangle} = C_x\left[\sqrt{n+1}^{\,2} + \sqrt{n}^{\,2}\right]\langle n|n\rangle$$

$$= C_x(n+1+n) = \underline{\frac{\hbar}{2m\omega}(2n+1)}$$

SIMILARLY FOR MOMENTUM

$\langle n|p|n\rangle$ :

$$p^2 = (i)^2 \frac{m\hbar\omega}{2}(a^+ - a)(a^+ - a)$$

$$= C_p(a^{+2} - a^+a - aa^+ + a^2)$$

AND SO (CONTRIBUTING TERMS ONLY)

$$\langle n|p^2|n\rangle = -\frac{m\hbar\omega}{2}\left(-\sqrt{n+1}^2 - \sqrt{n}^2\right)\langle n|n\rangle$$

$$= \frac{m\hbar\omega}{2}(2n+1) = m\hbar\omega(n+1/2)$$

$\langle n|x^2|n\rangle$ WAS $\frac{\hbar}{m\omega}(n+1/2)$

$$\langle(\Delta x)^2\rangle\langle(\Delta p)^2\rangle$$

$$\boxed{\begin{aligned} &= \langle x^2\rangle\langle p^2\rangle = \frac{\hbar}{\cancel{m\omega}}(n+1/2)\cancel{m}\hbar\cancel{\omega}(n+1/2) \\ &= (n+1/2)^2\hbar^2 \end{aligned}}$$

THE GENERALIZED UNCERTAINTY RELATION WAS :

$$\langle (\Delta A)^2 \rangle \langle (\Delta B)^2 \rangle \geq \frac{1}{4} \left| \langle [A,B] \rangle \right|^2$$

NOW $[x,p] = xp - px$

$$= \sqrt{\frac{\hbar}{2m\omega}} (a+a^\dagger)\, i\sqrt{\frac{m\hbar\omega}{2}} (a^\dagger - a)$$

$$- i\sqrt{\frac{m\hbar\omega}{2}} (a^\dagger - a) \sqrt{\frac{\hbar}{2m\omega}} (a+a^\dagger)$$

$$= \frac{i\hbar}{2}\left(aa^\dagger - \cancel{a^2} + \cancel{a^{\dagger 2}} - a^\dagger a - a^\dagger a - \cancel{a^{\dagger 2}} + \cancel{a^2} + aa^\dagger\right)$$

$$= \frac{i\hbar}{\cancel{2}} \cancel{2} (aa^\dagger - a^\dagger a) = i\hbar \underbrace{[a, a^\dagger]}_{=\mathbb{1}}$$

$$\Rightarrow [x,p] = i\hbar$$

WHICH WE KNEW ALREADY

THUS: THE COMMUTATOR EXPECTATION VALUE IS STATE INDEPENDENT

$$\langle [x, p] \rangle = \langle n | i\hbar | n \rangle = i\hbar$$

NOTICE FIXED NUMBER FOR FIXED STATE

AND

$$\frac{1}{4} \left| \langle [x, p] \rangle \right|^2 = \frac{\hbar^2}{4}$$

INSTEAD

$$\langle (\Delta x)^2 \rangle \langle (\Delta p)^2 \rangle = (n + 1/2)^2 \hbar^2$$

AND $(n + 1/2)^2 \hbar^2 \geq \frac{\hbar}{4}$

HOLDS $\forall n$

= SIGN HOLDS FOR THE GROUND STATE

AND

THE HIGHER THE EXCITATION

$\Rightarrow$ THE LARGER THE

$(n + 1/2)^2 \hbar^2$ TERM