Finally, we note that in considering the shift from free particles $[V(r)=0]$ to the case of the constant potential $V(r)=V_0$, we need only to replace the E in the free-particle solution [see (A.5.10) and (A.5.11) by $E-V_0$. Note, though, that if $E<V_0$, $h_l^{(1,2)}(i\kappa r)$ is to be used with $\kappa=\sqrt{2m(V_0-E)/\hbar^2}$.

A.6. HYDROGEN ATOM

Here the potential is

$$V(r) = -\frac{Ze^2}{r} \tag{A.6.1}$$

and we introduce the dimensionless variable

$$\rho = \left(\frac{8m_e|E|}{\hbar^2}\right)^{1/2} r. \tag{A.6.2}$$

The energy eigenfunctions and eigenvalues (energy levels) are

$$\psi_{nlm} = R_{nl}(r)Y_l^m(\theta,\phi)$$

$$R_{nl}(r) = -\left\{\left(\frac{2Z}{na_0}\right)^3 \frac{(n-l-1)!}{2n[(n+l)!]^3}\right\}^{1/2} e^{-\rho/2}\rho^l L_{n+l}^{2l+1}(\rho)$$

$$E_n = \frac{-Z^2e^2}{2n^2a_0} \qquad \text{(independent of } l \text{ and } m\text{)} \tag{A.6.3}$$

$$a_0 = \text{Bohr radius} = \frac{\hbar^2}{m_e e^2}$$

$$n \geq l+1, \qquad \rho = \frac{2Zr}{na_0}.$$

The associated Laguerre polynomials are defined as follows:

$$L_p^q(\rho) = \frac{d^q}{d\rho^q}L_p(\rho), \tag{A.6.4}$$

where—in particular—

$$L_p(\rho) = e^{\rho}\frac{d^p}{d\rho^p}(\rho^p e^{-\rho}) \tag{A.6.5}$$

and the normalization integral satisfies

$$\int e^{-\rho}\rho^{2l}\left[L_{n+l}^{2l+1}(\rho)\right]^2\rho^2\,d\rho = \frac{2n[(n+l)!]^3}{(n-l-1)!}. \tag{A.6.6}$$

The radial functions for low n are:

$$R_{10}(r) = \left(\frac{Z}{a_0}\right)^{3/2} 2e^{-Zr/a_0}$$

$$R_{20}(r) = \left(\frac{Z}{2a_0}\right)^{3/2} (2-Zr/a_0)e^{-Zr/2a_0} \tag{A.6.7}$$

$$R_{21}(r) = \left(\frac{Z}{2a_0}\right)^{3/2} \frac{Zr}{\sqrt{3}\,a_0}e^{-Zr/2a_0}.$$

The radial integrals are

$$\langle r^k\rangle \equiv \int_0^\infty dr\, r^{2+k}\,[R_{nl}(r)]^2,$$

$$\langle r\rangle = \left(\frac{a_0}{2Z}\right)[3n^2 - l(l+1)]$$

$$\langle r^2\rangle = \left(\frac{a_0^2n^2}{2Z^2}\right)[5n^2 + 1 - 3l(l+1)] \tag{A.6.8}$$

$$\left\langle\frac{1}{r}\right\rangle = \frac{Z}{n^2a_0},$$

$$\left\langle\frac{1}{r^2}\right\rangle = \frac{Z^2}{[n^3a_0^2(l+\frac{1}{2})]}.$$