Many fermion systems

We consider systems of identical non-relativistic fermions.

Noninteracting states

Let's look at the Hamiltonian

$$H_0 = \sum_{i=1}^{N} \left[-\frac{\hbar^2}{2m} \nabla_i^2 + V(\boldsymbol{r}_i) \right]$$

and the corresponding Schrödinger equation.

$$H_0 \Phi(\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_N) = E \Phi(\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_N). \qquad (*)$$

Assuming that the functions $f_i(\boldsymbol{r})$ satisfy the single particle equations

$$-\frac{\hbar^2}{2m} \nabla^2 f_i(\boldsymbol{r}) = \epsilon_i f_i(\boldsymbol{r}_i)$$

it is easy to see that

$$\Phi = f_1(\boldsymbol{r}_1) f_2(\boldsymbol{r}_2) \cdots f_N(\boldsymbol{r}_N) \qquad (**)$$

satisfies the many particle equation $(*)$ the eigenvalue being

$$E = \sum_{i=1}^{N} \epsilon_i .$$

However, the product $(**)$ is not a correct solution because

- spin degrees of freedom are not taken into account
- the statistics is wrong: the wave function describing identical fermions must be antisymmetric and the wave function describing bosons symmetric under the exhange of coordinates of two particles.

Spin coordinates

We employ the spin coordinates ξ which can take the values ± 1 and the spin wave functions $\chi_\uparrow$ and $\chi_\downarrow$:

$$\begin{aligned} \chi_\uparrow(\xi) &= \begin{cases} 1 & \text{if } \xi = 1 \\ 0 & \text{if } \xi = -1 \end{cases} \\ \chi_\downarrow(\xi) &= \begin{cases} 0 & \text{if } \xi = 1 \\ 1 & \text{if } \xi = -1. \end{cases} \end{aligned}$$

If there are no spin dependent terms in the Hamiltonian H_0 the one particle solutions can be written now in the form

$$\phi_i(\boldsymbol{x}) = f_i(\boldsymbol{r}) \chi_i(\xi),$$

where χ_i is either $\chi_\uparrow$ or $\chi_\downarrow$ and

$$\boldsymbol{x} = (\boldsymbol{r}, \xi).$$

Integration over the vector $\boldsymbol{x}$ includes also the summation over the variable ξ, i.e.

$$\int \phi_i^*(\boldsymbol{x})\phi_j(\boldsymbol{x})\, d\boldsymbol{x} = \sum_{\xi=\pm 1} \int f_i^*(\boldsymbol{r}) f_j(\boldsymbol{r})\, d\boldsymbol{r} \chi_i(\xi)\chi_j(\xi).$$

In particular we have

$$\int |\phi_i(\boldsymbol{x})|^2 d\boldsymbol{x} = \int |f_i(\boldsymbol{r})|^2 d\boldsymbol{r} \sum_{\xi=\pm 1} \chi_i^2(\xi) = 1,$$

provided that the space wave functions are normalized.

Fermions
We start with the product $\phi_1(\boldsymbol{x}_1)\phi_2(\boldsymbol{x}_2)\cdots\phi_N(\boldsymbol{x}_N)$ and

1. permute the coordinates $\boldsymbol{x}_i$ in all possible ways. There are $N!$ differen permutations.
2. exchange the sign always when a we swap a pair of coordinates.
3. sum all resulting terms.

When we denote

- permutations of the coordinates $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N$ with the symbol P,
- the number of exchanges needed for the permutation P with the symbol p,

we end up with the antisymmetric normalized wave function

$$\Phi = \frac{1}{(N!)^{1/2}} \sum_P (-1)^p P \phi_1(\boldsymbol{x}_1)\phi_2(\boldsymbol{x}_2)\cdots\phi_N(\boldsymbol{x}_N).$$

By the definition this can be written as the determinant

$$\Phi = \frac{1}{(N!)^{1/2}} \begin{vmatrix} \phi_1(\boldsymbol{x}_1) & \phi_1(\boldsymbol{x}_2) & \cdots & \phi_1(\boldsymbol{x}_N) \\ \phi_2(\boldsymbol{x}_1) & \phi_2(\boldsymbol{x}_2) & \cdots & \phi_2(\boldsymbol{x}_N) \\ \vdots & \vdots & & \vdots \\ \phi_N(\boldsymbol{x}_1) & \phi_N(\boldsymbol{x}_2) & \cdots & \phi_N(\boldsymbol{x}_N) \end{vmatrix}.$$

Determinantal wave functions
As eigenstates of a single particle Hamiltonian the wave functions $\phi_i(\boldsymbol{x})$ form a complete orthonormal set, i.e.

$$\sum_i \phi_i^*(\boldsymbol{x}')\phi_i(\boldsymbol{x}) = \delta(\boldsymbol{x} - \boldsymbol{x}').$$

and

$$\int \phi_i^*(\boldsymbol{x})\phi_j(\boldsymbol{x})\,d\boldsymbol{x} = \delta_{ij}.$$

Out of these we construct all possible states for N noninteracting identical fermions by

1. selecting N non equal integers $a_1, a_2, \ldots, a_N$, or equivalently from the infinite set
$$\phi_1, \phi_2, \ldots$$
N different wave functions
$$\phi_{a_1}, \phi_{a_2}, \ldots, \phi_{a_N},$$
in all possible ways
2. forming the determinants
$$\phi_a(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N) \\ = \frac{1}{(N!)^{1/2}} \sum_P (-1)^p P\phi_{a_1}(\boldsymbol{x}_1)\cdots\phi_{a_N}(\boldsymbol{x}_N).$$

Let

$$\phi_b(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N) \\ = \frac{1}{(N!)^{1/2}} \sum_P (-1)^p P\phi_{b_1}(\boldsymbol{x}_1)\cdots\phi_{b_N}(\boldsymbol{x}_N).$$

some other determinant.
Let $d\tau'$ stand for the volume element $d\boldsymbol{x}_1\cdots d\boldsymbol{x}_N$. The prime $'$ indicates summation over the spin coordinates.
We look at the N-body operator $F(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N)$. We say that F on *symmetric* if $PF = F$.
Evaluation of matrix elements and scalar products is greatly simplified by the *Lemma: If F is a symmetrical operatorn then*

$$\int \Phi_b^* F \Phi_a d\tau' = N^{1/2} \int\int \Phi_b^* F \phi_{a_1}(\boldsymbol{x}_1)\cdots\phi_{a_N}(x_N)d\tau'.$$

Proof: ...■
Using the lemma it is easy to show the determinantal wave functions are orthonormalized: due to the orthonormality of the single particle wave functions we can write

$$\int \Phi_a^*\Phi_a d\tau' \\ = (N!)^{1/2} \int \Phi_a^*\phi_{a_1}(\boldsymbol{x}_1)\cdots\phi_{a_N}(\boldsymbol{x}_N)\,d\tau'$$

$$
\begin{aligned}
&= \int \left[\sum_P (-1)^p P \phi^*_{a_1}(\boldsymbol{x}_1) \cdots \phi^*_{a_N}(\boldsymbol{x}_N) \right] \\
&\quad \times \phi_{a_1}(\boldsymbol{x}_1) \cdots \phi_{a_N}(\boldsymbol{x}_N)\, d\tau' \\
&= \int |\phi_{a_1}(\boldsymbol{x}_1)|^2 \cdots |\phi_{a_N}(\boldsymbol{x}_N)|^2\, d\tau' \\
&= \int |\phi_{a_1}(\boldsymbol{x}_1)|^2 d\boldsymbol{x}_1 \cdots \int |\phi_{a_N}(\boldsymbol{x}_N)|^2 d\boldsymbol{x}_N = 1.
\end{aligned}
$$

Let us suppose now that for the determinants Φ_a and Φ_b

$$
\begin{aligned}
a_i &= b_i \text{ when } i \neq j \\
a_j &\neq b_j
\end{aligned}
$$

holds, i.e. the determinants differ from eachother only in one single particle state. Then

$$
\begin{aligned}
&\int \Phi^*_a \Phi_b d\tau' \\
&\quad = \int \left[\sum_P (-1)^p P \phi^*_{a_1}(\boldsymbol{x}_1) \cdots \phi^*_{a_N}(\boldsymbol{x}_N) \right] \\
&\qquad \times \phi_{b_1}(\boldsymbol{x}_1) \cdots \phi_{b_N}(\boldsymbol{x}_N)\, d\tau' \\
&\quad = \int |\phi_{a_1}(\boldsymbol{x}_1)|^2 d\boldsymbol{x}_1 \cdots \int \phi^*_{a_j}(\boldsymbol{x}_j) \phi_{b_j}(\boldsymbol{x}_j)\, d\boldsymbol{x}_j \\
&\qquad \cdots \int |\phi_{a_N}(\boldsymbol{x}_N)|^2 d\boldsymbol{x}_N = 0.
\end{aligned}
$$

It is easy to see that this is true also when there are more than one differing state.
Thus the determinants Φ_a form an orthonormal set. It can also be shown that in the space of antisymmetric wave functions this set is complete.
Correspondingly any antisymmetric function Φ of N variables can be written as the superposition

$$
\Psi = \sum_a B_a \Phi_a.
$$

Matrix elements
The original Schrödinger equation

$$
H\Psi = E\Psi
$$

can be casted into the matrix form

$$
\sum_a (H_{ba} - E) B_a = 0, \qquad (*)
$$

where

$$H_{ba} = \langle \Phi_b | H | \Phi_a \rangle = \int \Phi_b^* (H_0 + H') \Phi_a d\tau'. \qquad (**)$$

Now

$$H_0 = \sum_{i=1}^{N} \left[-\frac{\hbar^2}{2m} \nabla_i^2 + V(\boldsymbol{r}_i) \right]$$

a sum of *single particle operators* (operators acting on a single particle at time) and

$$H' = \frac{1}{2} \sum_{i \neq j}^{N} v(\boldsymbol{r}_i, \boldsymbol{r}_j)$$

a sum of *two particle operators.*
The evaluation of the matrix elements H_{ba} involves thus working with one and two body operators.
Usually the operators are either one body operators

$$\Omega^{(1)} = \sum_{i=1}^{N} \eta(\boldsymbol{x}_i)$$

or two body operators

$$\Omega^{(2)} = \frac{1}{2} \sum_{i \neq j}^{N} \vartheta(\boldsymbol{x}_i, \boldsymbol{x}_j).$$

We employ the notations

$$\langle i | \eta | j \rangle = \int \phi_i^*(\boldsymbol{x}) \eta(\boldsymbol{x}) \phi_j(\boldsymbol{x}) \, d\boldsymbol{x},$$

and

$$\begin{aligned} &\langle ij | \vartheta | kl \rangle \\ &\quad = \iint \phi_i^*(\boldsymbol{x}_1) \phi_j^*(\boldsymbol{x}_2) \vartheta(\boldsymbol{x}_1, \boldsymbol{x}_2) \phi_k(\boldsymbol{x}_1) \phi_l(\boldsymbol{x}_2) \, d\boldsymbol{x}_1 d\boldsymbol{x}_2. \end{aligned}$$

In the case of fermions relative orders of indeces (i, j, k, l) and integration variables $(\boldsymbol{x}_1, \boldsymbol{x}_2)$ are essential in the latter definition.

One body operator
It can be shown that
(i): if $\Phi_a = \Phi_b$, i.e. $b_i = a_i \; \forall i$ then

$$\langle \Phi_a | \Omega^{(1)} | \Phi_b \rangle = \sum_{i=1}^{N} \langle a_i | \eta | a_i \rangle.$$

(ii): if Φ_a and Φ_b differ in one function, i.e.

$$\begin{aligned} a_i &= b_i \text{ when } i \neq k \\ a_k &\neq b_k, \end{aligned}$$

then

$$\langle\Phi_b|\Omega^{(1)}|\Phi_a\rangle = \langle b_k|\eta|a_k\rangle.$$

(iii): if Φ_a and Φ_b differ in two function, i.e.

$$\begin{aligned} a_i &= b_i \text{ when } i \neq k, l \\ a_k &\neq b_k, b_l \\ a_l &\neq b_k, b_l, \end{aligned}$$

then

$$\langle\Phi_b|\Omega^{(1)}|\Phi_a\rangle = 0.$$

This holds also when the determinants differ in more than two functions.
We see that
One body operators have nonzero matrix elements only between the determinants which differ at most in one single particle function.

Two body operator
It can be shown that
(i): if $\Phi_a = \Phi_b$ then

$$\langle\Phi_b|\Omega^{(2)}|\Phi_a\rangle = \frac{1}{2}\sum_{i\neq j}^{N}\left[\langle a_i a_j|\vartheta|a_i a_j\rangle - \langle a_j a_i|\vartheta|a_i a_j\rangle\right].$$

(ii): if Φ_a and Φ_b differ in one function, i.e.

$$\begin{aligned} a_i &= b_i \text{ when } i \neq k \\ a_k &\neq b_k, \end{aligned}$$

then

$$\begin{aligned} &\langle\Phi_b|\Omega^{(2)}|\Phi_b\rangle \\ &\quad = \sum_{i=1}^{N}\left[\langle a_i b_k|\vartheta|a_i a_k\rangle - \langle b_k a_i|\vartheta|a_i a_k\rangle\right]. \end{aligned}$$

(iii): if Φ_a and Φ_b differ in two functions, i.e.

$$\begin{aligned} a_i &= b_i \text{ when i!=k,l} \\ a_k &\neq b_k, b_l \\ a_l &\neq b_k, b_l, \end{aligned}$$

then

$$\begin{aligned} &\langle\Phi_b|\Omega^{(2)}|\Phi_b\rangle \\ &\quad = \langle b_k b_l|\vartheta|a_k a_l\rangle - \langle b_l a_k|\vartheta|a_k a_l\rangle. \end{aligned}$$

(iv): if Φ_a and Φ_b differ in more than two functions then

$$\langle\Phi_b|\Omega^{(2)}|\Phi_b\rangle = 0.$$

As an example lets look at the case *(ii)*, that is

$$\begin{aligned} a_i &= b_i \text{ when } i \neq k \\ a_k &\neq b_k. \end{aligned}$$

According to our previous lemma we have

$$\begin{aligned} &\langle\Phi_b|\Omega^{(2)}|\Phi_a\rangle \\ &\quad = \int \Phi_b^*\Omega^{(2)}\Phi_a d\tau' \\ &\quad = (N!)^{1/2}\int \Phi_b^*\Omega^{(2)}\phi_{a_1}(\boldsymbol{x}_1)\cdots\phi_{a_N}(\boldsymbol{x}_N)\,d\tau' \\ &\quad = \frac{1}{2}(N!)^{1/2}\sum_{i\neq j}^{N}\int \Phi_b^*\vartheta(\boldsymbol{x}_i,\boldsymbol{x}_j) \\ &\qquad\qquad \phi_{a_1}(\boldsymbol{x}_1)\cdots\phi_{a_N}(\boldsymbol{x}_N)\,d\tau'. \end{aligned}$$

Thus

$$\begin{aligned} &(N!)^{1/2}\int \Phi_b^*\vartheta(\boldsymbol{x}_i,\boldsymbol{x}_j)\phi_{a_1}(\boldsymbol{x}_1)\cdots\phi_{a_N}(\boldsymbol{x}_N)\,d\tau' \\ &\quad = \int\left[\sum_P(-1)^pP\phi_{a_1}^*(\boldsymbol{x}_1)\cdots\phi_{b_k}^*(\boldsymbol{x}_k)\cdots\phi_{a_N}^*(\boldsymbol{x}_N)\right] \\ &\qquad \vartheta(\boldsymbol{x}_i,\boldsymbol{x}_j)\phi_{a_1}(\boldsymbol{x}_1)\cdots\phi_{a_k}(\boldsymbol{x}_k)\cdots\phi_{a_N}(\boldsymbol{x}_N)\,d\tau' \\ &\quad = \begin{cases} \langle b_ka_j|\vartheta|a_ka_j\rangle - \langle a_jb_k|\vartheta|a_ia_k\rangle, & \text{when } i=k \\ \langle a_ib_k|\vartheta|a_ia_k\rangle - \langle b_ka_i|\vartheta|a_ia_k\rangle, & \text{when } j=k \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Now

$$\begin{aligned} &\langle ij|\vartheta|kl\rangle \\ &\quad = \int\int \phi_i^*(\boldsymbol{x}_1)\phi_j^*(\boldsymbol{x}_2)\vartheta(\boldsymbol{x}_1,\boldsymbol{x}_2)\phi_k(\boldsymbol{x}_1)\phi_l(\boldsymbol{x}_2)\,d\boldsymbol{x}_1d\boldsymbol{x}_2 \\ &\quad = \int\int \phi_i^*(\boldsymbol{x}_2)\phi_j^*(\boldsymbol{x}_1)\vartheta(\boldsymbol{x}_2,\boldsymbol{x}_1)\phi_k(\boldsymbol{x}_2)\phi_l(\boldsymbol{x}_1)\,d\boldsymbol{x}_1d\boldsymbol{x}_2 \\ &\quad = \langle ji|\vartheta|lk\rangle \end{aligned}$$

assuming that $\vartheta(\boldsymbol{x}_1,\boldsymbol{x}_2)$ is symmetric. So we get

$$\begin{aligned} &\langle\Phi_b|\Omega^{(2)}|\Phi_b\rangle \\ &\quad = \frac{1}{2}\sum_{j=1}^{N}\left[\langle b_ka_j|\vartheta|a_ka_j\rangle - \langle a_jb_k|\vartheta|a_ia_k\rangle\right] \end{aligned}$$

$$\sum_{i=1}^{N}[\langle a_i b_k|\vartheta|a_i a_k\rangle - \langle b_k a_i|\vartheta|a_i a_k\rangle]$$
$$= \sum_{i=1}^{N}[\langle a_i b_k|\vartheta|a_i a_k\rangle - \langle b_k a_i|\vartheta|a_i a_k\rangle] \blacksquare$$

We see that
Two body operators have nonzero matrix elements only between the determinants which differ at most in two single particle wave functions.

Creation ja annihilation operators
The sign of the determinantal wave function

$$\begin{aligned}&\Phi_{a_1 a_2 \ldots a_N}(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N)\\ &\quad = \frac{1}{(N!)^{1/2}} \sum_P (-1)^p P \phi_{a_1}(\boldsymbol{x}_1)\phi_{a_2}(\boldsymbol{x}_2)\cdots\phi_{a_N}(\boldsymbol{x}_N)\end{aligned}$$

depends on the order of the single particle wave functions ϕ_{a_i}. We remove this uncertainty by

- defining a linear order among the single particle states which will allow us to write
$$a_1 < a_2 \cdots < a_N.$$
In many cases the single particle energy together with the spin orientation are suitable quantities for the ordering.

- writing the single particle states in the determinantal wave functions always in the increasing order.

Annihilation operator
We define the annihilation (destruction) operator c_{a_k} formally so, that

$$\begin{aligned}&c_{a_k}\phi^N_{a_1\ldots a_k \ldots a_N}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N)\\ &\quad = \pm\Phi^{N-1}_{a_1\ldots a_{k-1}a_{k+1}\ldots a_N}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{N-1})\\ &\quad = \pm\frac{1}{[(N-1)!]^{1/2}} \sum_P (-1)^p P \phi_{a_1}(\boldsymbol{x}_1)\cdots\\ &\qquad\qquad \phi_{a_{k-1}}(\boldsymbol{x}_{k-1})\phi_{a_{k+1}}(\boldsymbol{x}_k)\cdots\phi_{a_N}(\boldsymbol{x}_{N-1}).\end{aligned}$$

Here the sign is

- *positive* if k is *odd*, i.e. in the determinant Φ^N the function ϕ_{a_k} is preceded by an even number of single particle states.

- *negative* if k is *even*, i.e. in the determinant Φ^N the function ϕ_{a_k} is prceded by an odd number of single particle states.

The destruction operator converts an N particle state containing the single particle state ϕ_{a_k} to a determinantal wave function of $N-1$ particles not containing the state ϕ_{a_k}.
If the determinant Φ^N does not contain the state ϕ_{a_k} we define

$$c_{a_k}\Phi^N_{a_1\ldots a_N} = 0 \text{ if } a_k \notin \{a_1, \ldots, a_N\}.$$

Creation operator
The creation operator $c_l^\dagger$ is defined formally so that

$$\begin{aligned}
&c_l^\dagger \Phi^N_{a_1\ldots a_j a_{j+1}\ldots a_N}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N) \\
&\quad = \pm\Phi^{N+1}_{a_1}\ldots a_j l a_{j+1}\ldots a_N)(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{N+1}) \\
&\quad = \pm\frac{1}{[(N-1)!]^{1/2}}\sum_P (-1)^p P\phi_{a_1}(\boldsymbol{x}_1)\cdots\phi_{a_j}(\boldsymbol{x}_j) \\
&\qquad\qquad \phi_l(\boldsymbol{x}_{j+1})\phi_{a_{j+1}}(\boldsymbol{x}_{j+2})\cdots\phi_{a_N}(\boldsymbol{x}_{N+1}).
\end{aligned}$$

Here $a_j < l < a_{j+1}$ and the sign is

- *positiive* if j is *even*, i.e. in the determinant Φ^{N+1} the function ϕ_l is preceded by an even number of single particle states.
- *negative* if j is *odd*, i.e. in the determinant Φ^{N+1} the function ϕ_l is preceded by an odd number of single particle states.

The creation operator converts an N particle state not containing the state ϕ_l to a determinantal $N+1$ particle state containing the state ϕ_l.
In the case the determinant Φ^N contains the state ϕ_l we define

$$c_l^\dagger\Phi^N_{a_1\ldots a_N} = 0 \text{ if } l \in \{a_1, \ldots a_N\}.$$

We should note that the sign associated with the

- annihgilation operator is defined to be positive if the single particle state is destroyed at the first position, i.e.

  $$c_{a_1}\Phi^N_{a_1 a_2\ldots a_N} = \Phi^{N-1}_{a_2\ldots a_N}.$$

  In other cases it corresponds to the number of swappings needed to move the state to be destructed to the first position, e.g.

  $$\begin{aligned}
  c_{a_2}\Phi^N_{a_1 a_2\ldots a_N} &= -c_{a_2}\Phi^N_{a_2 a_1\ldots a_N} \\
  &= -\Phi^{N-1}_{a_1 a_3\ldots a_N}.
  \end{aligned}$$

The sign associated with the

- creation operator is defined to be positive if we create a single particle state at the first position, i.e.

$$c_l^\dagger \Phi^N_{a_1 \ldots a_N} = \Phi^{N+1}_{l a_1 \ldots a_N}.$$

In other cases it corresponds to the number of swapping needed to move the created single particle state to its own place, e.g.

$$\begin{aligned} c_l^\dagger \Phi^N_{a_1 a_2 \ldots a_N} &= \Phi^{N+1}_{l a_1 a_2 \ldots a_N} \\ &= -\Phi^{N+1}_{a_1 l a_2 \ldots a_N}, \end{aligned}$$

if $a_1 < l < a_2$.

Occupation numbers

Instead of specifying a determinantal wave function Φ with indeces labeling the included single particle states we get a more convenient notation by telling the *occupation numbers* n_i of the single particle states ϕ_i:

- we let n_i to be 1 if the function ϕ_i is in the determinant Φ and 0 otherwise.
- we write the determinantal wave function of N particles as

$$\Phi^N(n_1, n_2, n_3, \ldots).$$

For example

$$\Phi^N_{a_1 a_2 \ldots a_N} \equiv \Phi^N(0_1, 0_2, \ldots, 1_{a_1}, \ldots, 1_{a_2}, \ldots, 1_{a_N}, \ldots).$$

An often used notation is also

$$|n_1, n_2, n_3, \ldots\rangle \equiv \Phi(n_1, n_2, n_3, \ldots),$$

or more compcatly only those indeces for which $n_i = 1$ are specified, i.e.

$$|0_1, 0_2, \ldots, 1_{a_1}, \ldots, 1_{a_2}, \ldots, 1_{a_N}, \ldots\rangle \equiv |a_1 a_2 \ldots a_N\rangle.$$

Employing the occupation number notation the creation and annihilation operators can be defined as

$$\begin{aligned} c_k \Phi^N(\ldots n_k \ldots) &= \theta_k n_k \Phi^{N-1}(\ldots 0_k \ldots) \\ c_k^\dagger \Phi^N(\ldots n_k \ldots) &= \theta_k (1 - n_k) \Phi^{N+1}(\ldots 1_k \ldots), \end{aligned}$$

where

$$\theta_k = (-1)^{\sum_{j<k} n_j}.$$

Starting from the definition one can verify the anticommutation rules

$$\begin{aligned} \{c_l, c_k\} &= 0 \\ \{c_k^\dagger, c_l^\dagger\} &= 0 \\ \{c_l^\dagger, c_k\} &= \delta_{kl}. \end{aligned}$$

With the help of these it is easy to see that

$$c_k^\dagger c_k \Phi^N(\ldots n_k \ldots) = n_k \Phi^N(\ldots n_k \ldots),$$

or in the operator fromalism

$$c_k^\dagger c_k = n_k.$$

Due to this the operator

$$N_k = c_k^\dagger c_k$$

is called *number operator*
We define the *vacuum* Φ_{vac} to be such a fictious "zeroth order" determinant that contains no single particle wave function, i.e.

$$\Phi_{\text{vac}} \equiv \Phi(0_1, 0_2, 0_3, \ldots).$$

Applying creation operators consecutively on to this we can construct any determinantal wave function. For example

$$\begin{aligned}
c_3^\dagger c_1^\dagger c_2^\dagger &= c_3^\dagger c_1^\dagger c_2^\dagger \Phi(0_1, 0_2, 0_3, \ldots) \\
&= c_3^\dagger c_1^\dagger \Phi^1(0_1, 1_2, 0_3, \ldots) \\
&= c_3^\dagger \Phi^2(1_1, 1_2, 0_3, \ldots) \\
&= (-1)^2 \Phi^3(1_1, 1_2, 1_3, 0_4, \ldots) \\
&= \frac{1}{(3!)^{1/2}} \begin{vmatrix} \phi_1(\boldsymbol{x}_1) & \phi_1(\boldsymbol{x}_2) & \phi_1(\boldsymbol{x}_3) \\ \phi_2(\boldsymbol{x}_1) & \phi_2(\boldsymbol{x}_2) & \phi_2(\boldsymbol{x}_3) \\ \phi_3(\boldsymbol{x}_1) & \phi_3(\boldsymbol{x}_2) & \phi_3(\boldsymbol{x}_3) \end{vmatrix}.
\end{aligned}$$

One and two body operators can be written with the help of creation and destruction operators in the form

$$\begin{aligned}
\Omega^{(1)} &= \sum_{i,j} \langle i|\eta|j\rangle c_i^\dagger c_j \\
\Omega^{(2)} &= \sum_{i,j,k,l} \langle ij|\vartheta|kl\rangle c_i^\dagger c_j^\dagger c_l c_k.
\end{aligned}$$