SCHWINGER'S OSCILLATOR MODEL

LET US CONSIDER 2 UNCOUPLED (INDEPENDENT) HARMONIC OSCILLATORS

FOR BOTH SEPARATELY

$$\boxed{N_i = a^\dagger_i a_i}$$

$$\boxed{[a_i, a^\dagger_i] = 1} \quad \boxed{[N_i, a_i] = -a_i}$$

$$\boxed{[N_i, a^\dagger_i] = a^\dagger_i}$$

AND

$$[a_1, a_2^\dagger] = [a_2, a_1^\dagger] = 0$$

AND OTHERS AS WELL

SINCE $[N_1, N_2] = 0$

$$\rightarrow \boxed{\begin{aligned} N_1 |n_1 n_2\rangle &= n_1 |n_1 n_2\rangle \\ N_2 |n_1 n_2\rangle &= n_2 |n_1 n_2\rangle \end{aligned}}$$

AND ALSO

$$a_1^\dagger |n_1 n_2\rangle = \sqrt{n_1+1}\,|n_1+1, n_2\rangle$$

$$a_2^\dagger |n_1 n_2\rangle = \sqrt{n_2+1}\,|n_1, n_2+1\rangle$$

$$a_1 |n_1 n_2\rangle = \sqrt{n_1}\,|n_1-1, n_2\rangle$$

$$a_2 |n_1 n_2\rangle = \sqrt{n_2}\,|n_1, n_2-1\rangle$$

STARTING FROM THE VACUUM, $|0,0\rangle$, WE GET A GENERAL EIGENKET OF N_1 & N_2 BY APPLYING REPEATEDLY THE CREATION OPERATORS:

$$\boxed{|n_1 n_2\rangle = \frac{(a_1^\dagger)^{n_1}(a_2^\dagger)^{n_2}}{\sqrt{n_1!}\sqrt{n_2!}}|0,0\rangle}$$

LET US DEFINE:

$$\boxed{J_+ = \hbar a_1^\dagger a_2} \quad \boxed{J_- = \hbar a_2^\dagger a_1}$$

$$\boxed{J_z = \left(\frac{\hbar}{2}\right)\left(a_1^\dagger a_1 - a_2^\dagger a_2\right) = \left(\frac{\hbar}{2}\right)\left(N_1 - N_2\right)}$$

THESE FOLLFILL THE ANGULAR MOMENTUM COMMUTATION RELATIONS

$$[J_z, J_\pm] = \pm \hbar J_\pm$$

$$\boxed{N = N_1 + N_2}$$

$$[J_+, J_-] = 2\hbar J_z$$

AND $\vec{J}^2 = \left(\frac{\hbar^2}{2}\right) N \left(\frac{N}{2} + 1\right)$

$$J_+|n_1 n_2\rangle = \hbar a_1^\dagger a_2 |n_1 n_2\rangle = \sqrt{n_2(n_1+1)}\,\hbar\,|n_1+1, n_2-1\rangle$$

$$J_-|n_1 n_2\rangle = \hbar a_2^\dagger a_1 |n_1 n_2\rangle = \sqrt{n_1(n_2+1)}\,\hbar\,|n_1-1, n_2+1\rangle$$

$$J_z|n_1 n_2\rangle = \left(\frac{\hbar}{2}\right)(N_1 - N_2)|n_1 n_2\rangle = \frac{\hbar}{2}(n_1 - n_2)|n_1 n_2\rangle$$

SUBSTITUTING

$n_1 \to j+m$

$n_2 \to j-m$

YIELDS TO THE

FAMILIAR FORMULAS

SO WHAT?

ASSOCIATE A SPIN-1/2 UP $(m=1/2)$

WITH OSCILLATOR #1

AND A SPIN-1/2 DOWN $(m=-1/2)$

WITH OSCILLATOR #2

$\to$ n_1 = # PARTICLES IN $|\uparrow\rangle$ STATE

n_2 = # PARTICLES $|\downarrow\rangle$ STATE

NOTICE $n_1 + n_2$ = CONSTANT

INSERTING

$$\boxed{j=\frac{n_1+n_2}{2}} \quad \boxed{m=\frac{n_1-n_2}{2}}$$

IN PLACE OF $|n_1 n_2\rangle$ TO CHARACTERIZE THE SIMULTANEOUS EIGENKETS OF $\vec{J}^2$ AND J_z

$$|jm\rangle=\frac{(a_1^{+})^{j+m}(a_2^{+})^{j-m}}{\sqrt{(j+m)!\,(j-m)!}}|0,0\rangle$$

AND THUS, AS FAR AS THE TRANSFORMATION PROPERTIES UNDER ROTATIONS ARE CONCERNED, WE CAN THINK OF ANY OBJECT WITH ANGULAR MOMENTUM j AS A COMPOSITION OF $2j$ SPIN $1/2$ PARTICLES

NOTICE, ABOVE METHOD WORKS FINE IF CONSIDER ROTATIONS OF SYSTEMS CHARACTERIZED BY j AND m AND IF WE ARE NOT INTERESTED HOW THESE STATES ARE ACTUALLY BUILD UP

FOR EXAMPLE

CONSIDER A STATE $j=1$, THIS IS COMPOSED OF 2 SPIN HALF PARTICLES WHICH ARE BOTH IN THE $|\uparrow\rangle$ STATE

$\longrightarrow$ VIOLATES PAULI PRINCIPLE