SOME NOTES ON UNITARY OPERATORS

- OUR $\mathcal{J}(d\bar{x})$ AND $U(t, t_0)$ ARE UNITARY

- LOOK AT TRANSF. $|\alpha\rangle \to U|\alpha\rangle$

$$\langle\beta|\alpha\rangle \to \langle\beta|U^{\dagger}U|\alpha\rangle = \langle\beta|\alpha\rangle$$

SCHRÖDINGER STYLE, U OPERATES ON BRAS & KETS

$$\langle\beta|X|\alpha\rangle = \left(\langle\beta|U^{\dagger}\right) X \left(U|\alpha\rangle\right)$$

$$= \langle\beta|U^{\dagger}XU|\alpha\rangle$$

$$\equiv \langle\beta|X'|\alpha\rangle \;; \quad X' = U^{\dagger}XU$$

I.E. PHYSICS IS THE SAME WHEN

- U OPERATES KETS

- SIMLARITY TRANSF. [redacted] FOR OBSERVABLES

EXAMPLE

$\boxed{\text{CASE 1}}$

TRANSLATION

- $|\alpha\rangle \to \mathcal{J}(\overline{dx}')|\alpha\rangle = \left(\mathbb{1} - \frac{i\bar{p}\cdot\overline{dx}'}{\hbar}\right)|\alpha\rangle$

- $X_{op} \to X_{op}$

OR $\boxed{\text{CASE 2}}$

$$X_{op} \to \left(\mathbb{1} + \frac{i\bar{p}\cdot\overline{dx}'}{\hbar}\right) X_{op} \left(\mathbb{1} - \frac{i\bar{p}\cdot\overline{dx}'}{\hbar}\right)$$

$$= X_{op} + \frac{i}{\hbar}\left[\bar{p}\cdot\overline{dx}', X_{op}\right] + \mathcal{O}(\overline{dx}'^2)$$

$$= X_{op} - \frac{\cancel{i}\cancel{i}\cancel{\hbar}\,\overline{dx}'}{\cancel{\hbar}}$$

$[p_{op}, X_{op}] = -i\hbar$

$$= X_{op} + \overline{dx}'$$

FROM EXERCISES CASE 1:

CASE 1 : $\langle x\rangle \to \langle x\rangle + \langle dx'\rangle$

CASE 2 : $\langle x\rangle \to \langle x\rangle + \langle dx'\rangle$

- WE HAVE AN OPERATOR A^S — TIME INDEPENDENT
SCHRÖDINGER PICTURE

- LET US DEFINE:

$$A^H(t) = U^\dagger(t) A^S U(t)$$

A^H IS HEISENBERG PICTURE OPERATOR AND $A^H(t=0) = A^S$

$$\frac{dA^H}{dt} = \frac{\partial}{\partial t} U^\dagger A^S U + U^\dagger A^S \frac{\partial U}{\partial t} \qquad (*)$$

REMEMBER SCHRÖDINGER EQUATION FOR U:

$$i\hbar \frac{\partial}{\partial t} U(t,t_0) = H U(t,t_0)$$

$$\Rightarrow \frac{\partial}{\partial t} U = \frac{H}{i\hbar} U \quad \text{AND} \quad \frac{\partial U^\dagger}{\partial t} = -\frac{1}{i\hbar} U^\dagger H$$

INSERT TO $(*)$

$$\frac{dA^H}{dt} = -\frac{1}{i\hbar} U^\dagger H A^S U + \frac{1}{i\hbar} U^\dagger A^S H U$$

$$= -\frac{1}{i\hbar} \left(U^\dagger H U\right)\left(U^\dagger A^S U\right) + \frac{1}{i\hbar} \left(U^\dagger A^S U\right)\left(U^\dagger H U\right)$$

$$= -\frac{1}{i\hbar} U^{+} H U A^{H} + \frac{1}{i\hbar} A^{H} U^{+} H U$$

$$= \frac{1}{i\hbar}\left[A^{H}, U^{+} H U\right]$$

DO WE DARE TO IDENTIFY

$$H^{H} = U^{+} H^{S} U \quad ?$$

SURE WE DO AS LONG AS

$$U = e^{-iHt/\hbar} \rightarrow [U, H] = 0$$

$$\rightarrow U^{+} H U = H$$

AND SO

$$\boxed{\frac{dA^{H}}{dt} = \frac{1}{i\hbar}\left[A^{H}, H\right]}$$

WHICH IS THE

<u>HEISENBERG EQUATION OF MOTION</u>

<u>NOTICE</u> IF $\left[A^{H}(t), H(t)\right] = 0$

THEN $\frac{dA^{H}(t)}{dt} = 0$ A^{H} INDEPENDEN[T] ON TIME

FREE PARTICLE IN HEISENBERG PICTURE

- CLASSICAL ANALOQUE?
  CERTAINLY

$$\Rightarrow H^H_{op} = \frac{\bar{p}^{H\,2}_{op}}{2m} = \frac{\left(p^{H\,2}_{xop} + p^{H\,2}_{yop} + p^{H\,2}_{zop}\right)}{2m}$$

DROP op AND H FROM NOW ON

KEEP IN MIND OPERATORS ARE

TIME DEPENDENT

$$\frac{dp_i}{dt} = \frac{1}{i\hbar}[p_i, H] = 0$$

$$\Rightarrow p_i(t) = p_i(0) = \text{CONSTANT}$$

$$\frac{dx_i}{dt} = \frac{1}{i\hbar}[x_i, H] = \frac{1}{i\hbar}\left[x_i, \frac{\bar{p}^2}{2m}\right]$$

$$= \frac{1}{i\hbar 2m}[x_i, \bar{p}^2] \overset{\circledast}{=} \frac{1}{2m}\frac{\partial}{\partial p_i}\left(p_x^2 + p_y^2 + p_z^2\right)$$

$$= \frac{2p}{2m} = \frac{p_i}{m} = \underline{\underline{\frac{p_i(0)}{m}}}$$

$\circledast$ CAN BE SHOWN THAT

$$\boxed{[x_i, f(\bar{p})] = i\hbar\frac{\partial f(\bar{p})}{\partial p_i}} \quad \boxed{[p_i, f(\bar{x})] = -i\hbar\frac{\partial f(\bar{x})}{\partial x_i}}$$

p6

AND SO

$$X_i(t) = X_i(0) + \frac{p_i(0)}{m} t$$

IMPORTANT TO NOTICE:

$$[X_i(0), X_j(0)] = 0$$

BUT

$$[X_i(t), X_i(0)] = \left[X_i(0) + \frac{p_i(0)}{m} t, X_i(0)\right]$$

$$\frac{t}{m}[p_i(0), X_i(0)] = -\frac{i\hbar t}{m} \neq 0 \ !$$

UNCERTAINTY RELATION IS

$$\langle(\Delta X_i(t))^2\rangle\langle(\Delta X_i(0))^2\rangle \geq \frac{|\langle[X_i(t), X_i(0)]\rangle|^2}{4}$$

$$= \frac{\hbar^2 t^2}{4m^2}$$

AND SO, FOR A FREE PARTICLE, EVEN IF THE ~~PARTICLE THE~~ PARTICLE POSITION IS WELL KNOWN AT $t=0$, THE UNCERTAINTY INCREASES AS TIME PASSES

A PARTICLE IN A POTENTIAL V

$$H = \frac{\bar{p}^2}{2m} + V(\bar{x})$$

$\bar{x}$ = OPERATOR
$\bar{p}$ = OPERATOR

$$\frac{dp_i}{dt} = \frac{1}{i\hbar}[p_i, V(\bar{x})] = -\frac{\partial}{\partial x_i} V(\bar{x})$$

$$\frac{dx_i}{dt} = \frac{1}{i\hbar}[x_i, H] = \frac{p_i}{m}$$

since
$[p_i, \bar{p}^2] = 0$
$[x_i, V(\bar{x})] = 0$

AND

$$\frac{d^2x_i}{dt^2} = \frac{1}{i\hbar}\left[\frac{dx_i}{dt}, H\right] = \frac{1}{i\hbar}\left[\frac{p_i}{m}, H\right]$$

$$= \frac{1}{i\hbar m} i\hbar \frac{dp_i}{dt} = \frac{1}{m}\frac{dp_i}{dt} = -\frac{1}{m}\frac{\partial}{\partial x_i} V(\bar{x})$$

OR MORE GENERALLY

$$\boxed{m\frac{d^2\bar{x}}{dt^2} = -\nabla V(\bar{x})}$$

COMPARE WITH CLASSICAL ANALOGY

$m\bar{a} = \bar{F}$

THIS HOLDS FOR HEISENBERG PICTURE

IF WE TAKE EXPECTATION VALUE (IN H-PICTURE $|\alpha, t\rangle = |\alpha, 0\rangle$)

$$\rightarrow m \frac{d^2}{dt^2} \langle \bar{x} \rangle = \frac{d \langle \bar{p} \rangle}{dt} = - \langle \nabla V(\bar{x}) \rangle$$

= EHRENFEST THEOREM AND IS VALID BOTH FOR SCHRÖDINGER AND HEISENBERG PICTURES

SCHRÖDINGER EQUATION

- WE ARE IN THE POSITION TO DERIVE SCHRÖDINGER EQUATION

- CHOOSE $H = \frac{\bar{p}^2}{2m} + V(\bar{x})$

WHERE $V(\bar{x})$ IS HERMITEAN (AND LOCAL): $\langle x'' | V(\bar{x}) | x' \rangle = V(x')\,\delta(x'' - x')$

WE KNOW: $i\hbar \frac{\partial}{\partial t} |\alpha, t\rangle = H |\alpha, t\rangle$ (1)

$\bar{p} = -i\hbar \nabla$ (2)

MULTIPLY (1) FROM LEFT WITH $\langle x' |$

$$\Rightarrow i\hbar \frac{\partial}{\partial t} \langle x' | \alpha, t\rangle = -\frac{\hbar^2}{2m} \nabla^2 \langle x' | \alpha, t\rangle + V(x') \langle x' | \alpha, t\rangle$$

(*)
$$\Rightarrow \boxed{i\hbar \frac{\partial}{\partial t} \psi_\alpha(x', t) = -\frac{\hbar^2}{2m} \nabla^2 \psi_\alpha(x', t) + V(x') \psi_\alpha(x', t)}$$

IF THE SYSTEM IS INITIALLY IN AN EIGENSTATE OF OPERATOR A FOR WHOM $[A, H] = 0$ THEN

$$U|a', t\rangle = e^{-iE_a t/\hbar} |a', 0\rangle$$

$$\to \langle x' | a', t\rangle = e^{-iE_a t/\hbar} \psi_{a'}(x')$$

AND SO $i\hbar \frac{\partial}{\partial t} e^{-iE_a t/\hbar} \psi_{a'}(x')$

$$= E_a e^{-iE_a t/\hbar} \psi_{a'}(x')$$

AND

PLUGIN $\langle x', a', t\rangle = e^{-iE_a t/\hbar} \psi_{a'}(x')$

INTO $(\ast)$

$$\Rightarrow \boxed{E_a \psi_{a'}(x') = -\frac{\hbar^2}{2m} \nabla^2 \psi_{a'}(x') + V(x') \psi_{a'}(x')}$$

= TIME-INDEPENDENT SCH-E