Mathematical Methods

Lecture 1.
15/9-2008

1 Complex Numbers

Complex numbers are a generalization of the real numbers. We write

$$z \in \mathbb{C}, \qquad z = x + iy \qquad x, y \in \mathbb{R}, \quad i^2 = -1.$$

In analogy to the *real line* we speak of the *complex plane*, denoting a complex number by coordinates $(\mathrm{Re}(z), \mathrm{Im}(z))$.

Mathematical operations include

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2), \quad z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1).$$

The complex numbers with these operations are a *field*, which means that arithmetic works as for the real numbers. We define *complex conjugation* and the *modulus*

$$z = x + iy \to z^* = x - iy, \qquad |z| = \sqrt{z^* z} = \sqrt{x^2 + y^2}.$$

Then we can write

$$z = |z| \left(\cos(\theta) + i \sin(\theta)\right) = |z| e^{i\theta},$$

where θ is the *phase* or the *argument* $\theta = \arg(z)$.

Assuming that the complex exponential function works as the real one (which it does, see below), we have (De Moivre's Theorem)

$$z_1 z_2 = |z_1||z_2| e^{i(\theta_1 + \theta_2)}, \qquad \sqrt{|z| e^{i\theta}} = \sqrt{|z|} e^{i\theta/2}.$$

Ex. $z_1 = 2 + i$, $z_2 = 1 - 3i$.

$$z_1 + z_2 = 3 - 2i, \qquad z_1 z_2 = 5 - 5i, \qquad |z_1| = \sqrt{5}, \qquad z_1/z_2 = \frac{1}{2}(1 - i), \qquad \arg(z_1) = \tan^{-1}(1/2).$$

1.1 Application 0

A polynomial of n'th order has exactly n complex roots (counted with multiplicity), ie solutions z_j, $j = 1..n$ to

$$P_n(z) = a_0 z^n + a_1 z^{n-1} + ... + a_n = 0$$

and the polynomial can be written as

$$P_n(z) = a_0 (z - z_1)(z - z_2)...(z - z_n).$$

Ex. $P_2(z) = 2z^2 + (4 - 2i)z - 4i = 0$. $z_1 = i$, $z_2 = -2$.

What does it mean?
Complex numbers work like reals, except that they are not ordered along a line. Remembring that $i^2 = -1$, operations are like "multiplying out paranthesis" $(x + iy)$. Complex numbers have a real and an imaginary part, or equivalently a modulus and an argument.

2 Complex functions

Consider a complex functions of complex variables, $f : \mathbb{C} \to \mathbb{C}$. We can think of these in terms of real functions of real variables,

$$f(z) \to u(x, y) + iv(x, y), \qquad f = u + iv, \quad z = x + iy \quad u, v, x, y \in \mathbb{R}.$$

It is also called a *mapping* or a *tranformation*.

3 Basic topology

Consider subsets B of the complex plane.

A *neighbourhood* of a point z_0 is the set $N_\epsilon(z_0) = \{z \in \mathbb{C} : |z - z_0| < \epsilon\}$, for some ϵ.

A set B is *open* if for every point z in B, there exists a neighbourhood of z included in B.

A set is *closed* if its complement (in $\mathbb{C}$) is open.

A point z_0 is a *limit* point of B if every neighbourhood of z_0 includes a point in B.

For a function $f : B \to \mathbb{C}$ and z_0 a limit point of B, we say that w_0 is the *limit* of f as z goes to z_0, $\lim_{z \to z_0} f(z)$, if

$$\forall \epsilon, \quad \exists \delta \text{ ,so that } \; f(N_\delta(z_0)) \subset N_\epsilon(w_0).$$

Limits are unique and obey the usual properties,

$$\begin{aligned}
\lim_{z \to z_0} (f(z) \pm g(z)) &= \lim_{z \to z_0} f(z) \pm \lim_{z \to z_0} g(z), \\
\lim_{z \to z_0} (f(z) g(z)) &= \left(\lim_{z \to z_0} f(z)\right)\left(\lim_{z \to z_0} g(z)\right), \\
\lim_{z \to z_0} (f(z)/g(z)) &= \left(\lim_{z \to z_0} f(z)\right) / \left(\lim_{z \to z_0} g(z)\right).
\end{aligned}$$

In particular, for a complex function $f = u + iv$ of $z = x + iy$,

$$\lim_{z \to z_0} f(z) = w_0 \leftrightarrow \left[\lim_{z \to z_0} u(x, y) = \mathrm{Re}(w_0) \;\; \text{and} \;\; \lim_{z \to z_0} v(x, y) = \mathrm{Im}(w_0)\right].$$

3.1 Continuity

A function $f : B \to \mathbb{C}$ is *continuous* at a point z_0 if

$$\lim_{z \to z_0} f(z) = f(z_0).$$

It is said to be continuous in a set B if it is continuous for all $z \in B$.

Continuous functions have the usual properties: if f_1 and f_2 are continuous over B, so are $f_1 \pm f_2$, $f_1 f_2$ and f_1/f_2 ($f_2 \neq 0$). Also the composition $f(g(z))$ is continuous. We have for a complex function $f = u + iv$ of $z = x + iy$

$$f(z) \text{ continuous} \leftrightarrow [u(z) \;\; \text{and} \;\; v(z) \;\; \text{continuous}].$$

A *path* $s(t)$ is a continuous function from an interval $t \in [a, b] \subset \mathbb{R}$ into $\mathbb{C}$.

Ex. Straight line: $s(t) = z_0 + ct$, $z_0, c \in \mathbb{C}$, $t \in [a, b]$.

Ex. Circle: $s(t) = z_0 + re^{it}$, $t \in [0, 2\pi]$, $r > 0$.

Paths can be generalized to combinations of path segments, glued together at the ends. A set B is *path connected* if any two points in B can be connected by a path completely in B. A *domain* is a path connected open subset of the complex plane.

What does it mean?
Complex functions are defined like real ones, except that they map a plane onto a plane. At this point, topology and continuity works as for real functions. But beware of points where the functions are not defined, for one reason or another. These will become crucial later on. So will paths, which will take the role that the real axis usually has for "integrating along".

4 Power series

A complex *sequence* is a mapping $f : \mathbb{N} \to \mathbb{C}$. We write $f(n) = z_n$, $n = 0, 1, 2, ...$ A sequence is *convergent* and has the limit z_l if $\forall\epsilon$, $\exists N$, so that $\forall n > N$, $z_n \in N_\epsilon(z_l)$. We then write $\lim_{n\to\infty} z_n = z_l$. We have, with $z_n = x_n + iy_n$ and $z_l = x_l + iy_l$,

$$\lim_{n\to\infty} z_n = z_l \leftrightarrow \left[\lim_{n\to\infty} x_n = x_l \text{ and } \lim_{n\to\infty} y_n = y_l\right].$$

A sequence that is not convergent is *divergent*.

Theorem A sequence is convergent if and only if it is a *Cauchy sequence:*: Meaning that $\forall\epsilon$ $\exists N$, so that $\forall m, n > N \to |z_n - z_m| < \epsilon$.

Ex. $z_n = i\sqrt{2} + \left(\frac{3-4i}{6}\right)^n$. $|z_m - z_n|... < 2\left(\frac{5}{6}\right)^{min(n,m)} \to 0$.

From a sequence we can construct the partial sums $s_n = \sum_{m=0}^{n} z_m$. This defines a new sequence s_n. If it has a limit s_l, we say that we have a convergent *series*

$$s_l = \sum_{m=0}^{\infty} z_m = \lim_{n\to\infty} \sum_{m=0}^{n} z_m.$$

We of course have

$$s_l = x_l + iy_l = \sum_{m=0}^{\infty} x_m + i \sum_{m=0}^{\infty} y_m,$$

and ($c \in \mathbb{C}$)

$$\sum_{m=0}^{\infty}(z_m^1 + z_m^2) = \sum_{m=0}^{\infty} z_m^1 + \sum_{m=0}^{\infty} z_m^2, \qquad c\sum_{m=0}^{\infty} z_m = \sum_{m=0}^{\infty} cz_m.$$

If $\sum |z_m|$ is convergent, we say that $\sum z_m$ is *absolutely convergent*. Absolute convergence implies normal convergence.

Theorem: If $\sum z_n^1 = z_l^1$ and $\sum z_n^2 = z_l^2$ are absolutely convergent, then $\sum z_n^1 \sum z_n^2 = z_l^1 z_l^2$ is convergent.

Ex. $\sum \frac{i^n}{n^2}$ is convergent, since it is absolutely convergent. $\sum |z_n| = \frac{1}{n^2} = \frac{\pi^2}{6}$.

4.1 Convergence tests

Comparison test: If $\sum_m z_m^1$ is absolutely convergent, and $|z_m^1| > r|z_m^2|$ for $m > N$, for some N and $r > 0$, then $\sum z_m^2$ is absolutely convergent.

Ratio test: Consider the case when $\lim_{m\to\infty} \frac{|z_m|}{|z_{m-1}|} = \lambda$. If $\lambda < 1$ the series $\sum_{m=0}^{\infty} z_m$ is absolutely convergent. If $\lambda > 1$ it is divergent. If $\lambda = 1$ we don't know.

We define a *power series* of z around $z_0 \in \mathbb{C}$ with coefficients a_n by

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

Theorem: If a power series converges for $z = z_1$, then it converges absolutely for all z with $|z - z_0| < |z - z_1|$. If it diverges for $z = z_1$, then is diverges for all z with $|z - z_0| > |z - z_1|$. The *radius of convergence* R is the supremum of $|z|$ for the z for which the series converges. The series is convergent for $|z| < R$ and divergent for $|z| > R$. R can be zero or ∞.

For $\sum a_m (z - z_0)^m$, the radius of convergence is

$$R = \lim_{m\to\infty} |a_{m-1}/a_m|$$

if it exists. In general

$$\frac{1}{R} \limsup |a_m|^{1/m}.$$

Ex. The series $\sum z^n$ has $R = \lim_{m\to\infty} 1 = 1$.
Ex. The series $\sum \frac{z^n}{n}$ has $R = \lim_{m\to\infty} n = \infty$

We define

$$\exp(z) = \sum \frac{1}{n!} z^n, \quad \cos(z) = \sum \frac{(-1)^n}{(2n)!} z^{2n}, \quad \sin(z) = \sum \frac{(-1)^n}{(2n+1)!} z^{2n+1}.$$

Then we have the Euler formula

$$\exp(i\theta) = \cos(\theta) + i\sin(\theta),$$

and, as advertised, $z = |z| e^{i \arg(z)}$. More on $\arg(z)$ later.

What does it mean?
It will turn out that a function has a power series expansion in some region around every point where it is differentiable, the size of which is given by the radius of convergence. Therefore, manipulation of series is very useful in dealing with complex functions. The same function can have many power series, expanded around different points, and these may have different radii of convergence. More about this later.

5 Differentiation

A function $f : B \to \mathbb{C}$ is *differentiable* at z_0 if

$$\lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0)$$

exists and is the same when the limit is taken along any curve ending at z_0. Higher derivatives are defined in a similar way.

If f is differentiable at all points in (the open set) B we say that it is differentiable or *analytic* or *holomorphic* or *regular* on B. Differentiable functions are continuous.

Algebraic operations combine with differentiation in the usual way,

$$(f \pm g)' = f' \pm g', \quad (fg)' = f'g + g'f \quad (f/g)' = (f'g - g'f)/g^2.$$

The chain rule applies,

$$(g(f(z_0)))' = g'(f(z_0))f'(z_0).$$

This also applies for complex functions of real variables, such as paths.

Theorem: For $f(x+iy) = u(x,y) + iv(x,y)$, the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y},$$

hold and all the partial derivatives are continuous, if and only if $f(z)$ is differentiable.

Ex. $f(z) = |z|^2$.

$$u(x,y) = x^2 + y^2, \qquad v(x,y) = 0 \qquad \to \qquad \partial_x u = 2x, \partial_y u = 2y, \partial_x v = \partial_y v = 0.$$

Cauchy-Riemann equations are only satisfied and the derivatives continuous at $z = 0$. Hence f is differentiable there and nowhere else.

The real and imaginary parts $u(x,y), v(x,y$ are harmonic functions, obeying Laplace's equation

$$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = 0 = \frac{\partial^2 v(x,y)}{\partial x^2} + \frac{\partial^2 v(x,y)}{\partial y^2}.$$

Theorem: If f is differentiable in a domain D and $f'(z) = 0$ everywhere in D, f is a constant. If Ref, Imf or $|f|$ is constant, f is constant.

Theorem Within the radius of convergence, $|z - z_0| < R$ (where the series is absolutely convergent), we can take the derivative of a power series term by term,

$$\left(\sum a_n (z - z_0)^n\right)' = \sum n a_n (z - z_0)^{n-1},$$

and the result is convergent. This applies also to any higher order derivative. Hence

$$a_n = \frac{f^{(n)}(z_0)}{n!}.$$

where $f^{(n)}$ denotes differentiation n times. This shows that the power series is in fact the Taylor series,

$$f(z) = \sum \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n.$$

Turns out *all* complex differentiable functions can be expressed as power series, and infinitely differentiable and integrable. More about this later.

Ex. $f(z) = \frac{1}{1-z} = \sum_n z^n$. The series has radius of convergence $R = 1$. Then

$$f'(z) = \sum_n n z^{n-1} = \frac{1}{(1-z)^2}.$$

Taylor expansion around $z_0 = 0$,

$$f(0) = 1, \quad f'(0) = 1, \quad f''(0) = 2, \quad f^{(3)}(0) = 6, \quad f^{(4)}(0) = 24, \quad ... \quad f^{(n)}(0) = n!.$$

$$f(z) = \sum_n \frac{f^{(n)}}{n!}(z - z_0)^n = \sum_n z^n.$$

Around $z_0 = 2$?

$$f^{(n)} = n!(-1)^{n+1}, \qquad f(z) = \sum_n \frac{f^{(n)}}{n!}(z - z_0)^n = \sum_n (-1)^{n+1}(z-2)^n, \qquad R = \lim_{n\to\infty} 1 = 1.$$

Around $z_0 = -1$?

$$f^{(n)} = \frac{n!}{2^{n+1}}, \qquad f(z) = \sum_n \frac{f^{(n)}}{n!}(z - z_0)^n = \sum_n \frac{1}{2^{n+1}}(z+1)^n, \qquad R = \lim_{n\to\infty} 2 = 2.$$

What does it mean?
The definition of differentiability restricts functions to a fairly small subset, represented by the Cauchy-Riemann equations. On the other hand, once a function is known to be complex differentiable, we also know that it has a power series expansion, is infinitely many times differentiable and that the real and imaginary parts are harmonic funtions. Differentiation can for instance be done term by term in the power series, which turns out to be the Taylor series.