Mathematical Methods

Lecture 10.
24/11-2008

1 The concept of a Green's function

Consider the standard solution to a linear second order ODE,

$$a(x)y''(x) + b(x)y'(x) + c(x) = r(x), \qquad y(x) = y_h(x) + y_p(x),$$

where the homogeneous set of solutions is

$$y_h(x) = c_1 y_1(x) + c_2 y_2(x),$$

in terms of two linearly independendent solutions y_1, y_2, and the particular solution

$$y_p(x) = -y_1(x) \int \frac{y_2(x')r(x')}{W(x')} dx' + y_2(x) \int \frac{y_1(x')r(x')}{W(x')} dx',$$

with the Wronskian $W(x) = y_1(x)y_2'(x) + y_2(x)y_1'(x)$. We can write this as

$$y_p(x) = \int G(x,x')r(x')dx', \qquad G(x,x') = -\frac{y_1(x)y_2(x')}{W(x')} + \frac{y_2(x)y_1(x')}{W(x')},$$

and we call $G(x,x')$ the *Green's function* for the differential operator.
The Greens function is a *two-point function*, and viewed as a function of its first argument, it solves the homogeneous differential equation

$$D(x) = \left(a(x)\frac{d^2}{dx^2} + b(x)\frac{d}{dx} + c(x) \right) G(x,y) = 0, \qquad x \neq y.$$

It is continuous everywhere, also at $x = y$. Its derivative is discontinuous at $x = y$ with a jump

$$G'(x,x^+) - G'(x,x^-) = -1.$$

It is symmetric $G(x,y) = G(y,x)$. It satisfies the boundary condition

$$G(x,0) = G(x,l) = 0.$$

But we also have

$$u(y) = \int_0^l f(x)G(x,y)dx = \int_0^l (-D(x)U(x))G(x,y)dx = +\int_0^l u(x)(D(x)G(x,y))dx$$

which means that

$$D(x)G(x,y) = \delta(x-y).$$

Or, think of a general linear operator L acting on vectors u, f in a vector space V,

$$Lu = -f \quad \rightarrow \quad u = -L^{-1}f,$$

in terms of the inverse operator $LL^{-1} = 1$.

2 Finite dimension and linear operators

Consider a finite dimensional vector space V of dimension n. Any set of n linearly independent vectors v_i forms a *basis* for the vector space, and we can write for any other vector u,

$$u = \sum_i c_i v_i.$$

Let's specialise to an *inner product space*, with an "dot" product (u, v), which maps two vectors to a complex number. It is linear in the first variable

$$(au_1 + bu_2, v) = a(u_1, v) + b(u_2, v),$$

and conjugate linear in the second variable

$$(u, av_1 + bv_2) = a^*(u, v_1) + b^*(u, v_2).$$

Then we can define the *norm* of a vector

$$|v| = \sqrt{(v, v)}.$$

We can also define orthogonal vectors to be u and v, for which

$$(u, v) = 0.$$

One can always find a basis of vectors, which is at the same time orthogonal and has norm 1, ie is orthonormal. Assume e_i, $i = 0, .., n$ is such a basis.

Then for any vector, we have

$$v = \sum_i c_i e_i, \qquad c_i = (v, e_i), \quad \rightarrow \quad v = \sum_i (v, e_i) e_i.$$

In this way, we can define an *isomorphism* between V and the Euclidean, complex vector space C^n by

$$v \quad \rightarrow \quad (c_1, c_2, c_3, ..., c_n).$$

So in this sense *the only finite dimensional inner prodct spaces are* C^n.

For *infinite dimensional* inner product spaces, all of the above holds, except the isomorphism mapping. Infinite dimensional vector spaces have a more varied structure. They include: a) $\mathcal{C}$, the space of continuous functions, b) $\mathcal{C}^1$, the space of continuously differentiable functions, c) $\mathcal{C}^\infty$ the space of infinitely many times differentiable (or analytic) functions, d) $L_2[a, b]$, the space of square integrable functions on some interval $[a, b]$.

2.1 Linear transformations

A linear transformation or *operator* is a mapping L from V to V for which we have

$$L(au_1 + bu_2) = aL(u_1) + bL(u_2).$$

With $u = Lv$, we have that

$$u = \sum_i d_i e_i = L \sum_j c_j e_j = \sum_j c_j L(e_j)$$

And so

$$d_i = \sum_j c_j (L(e_j), e_i),$$

which is the Euclidean matrix-vector equation

$$u_i = M_{ij} v_j, \qquad M_{ij} = (L(e_j), e_i).$$

In other words, to describe any linear transformation, we only need to know how it acts on a set of basis vectors. This applies to finite and infinite dimensional spaces alike.

A linear operator L is said to be *bounded* if there is an M so that for all vectors v,

$$|L(v)| < M|v|.$$

Bounded linear operators constitute a vector space themselves. There is a one-to-one mapping between bounded linear operators on $\mathcal{R}^n$ and the space of n-dimensional matrices, even for n going to infinity.

2.2 Eigenvalues and eigenvectors

An operator may have eigenvectors u_n and eigenvalues λ_n, with

$$L(u_n) = \lambda_n u_n.$$

Eigenvectors with different eigenvalues are mutually linearly independent, and so V splits up into sub-spaces of lower dimensions, one for each distinct eigenvalue. The dimensionalities of these subspaces adds up to n. As a consequence, if all eigenvalues are distinct, the eigenvectors are all linearly independent, and therefore constitute a basis for V.

An operator L has an *inverse* L^{-1}, if and only if it has no zero eigenvalues

$$L(u) = 0, \quad \rightarrow \quad u = 0.$$

This is the analogue of a matrix being invertible only if it has non-zero determinant. Remember that the determinant of a matrix is the product of its eigenvalues.

Ex.: The matrix

$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

has eigenvalues $1 \pm i$ and eigenvectors $(1, i)$, $(1, -i)$. These are linearly independent, but not orthogonal, so they are a basis for the space $\mathcal{R}^2$. The determinant is 2, and so the inverse exists and is

$$\frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

An operator L on a real vector space is *symmetric* if

$$(L(v), u) = (v, L(u)).$$

For a complex vector space, the analogue is a *Hermitian* or *self-adjoint* operator, which has

$$(L(v), u) = (v, L(u))^*.$$

If L is symmetric/Hermitian/self-adjoint, the eigenvectors or distinct eigenvalues are orthogonal, and so V has an orthonormal basis of eigenvectors of L. The eigenvalues are real.

Ex.: The symmetric matrix

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

has real eigenvalues $\pm\sqrt{2}$, and eigenvectors $(1, 1-\sqrt{2})$, $(1, 1+\sqrt{2})$. These are orthogonal (and can easily be normalised), and so form a basis for the space R^2. The inverse matrix exists and is

$$\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Ex.: Consider a symmetric operator with n distinct eigenvalues. Then use the set of eigenvectors as the basis, and we have

$$L(v) = \sum_i c_i L(e_i) = \sum_i (v, e_i)\lambda_i e_i.$$

Ex.: Consider the operator that projects on the basis vector e_1,

$$L(v) = \sum \lambda_i (v, e_i) e_i, \qquad \lambda_i = \delta_{i1}.$$

It is linear and is written in terms of its eigenvectors. There are two distinct eigenvalues 1 and 0 (many times...).

3 Integral and differential operators

For our purposes, we should think of infinite dimensional vector spaces as spaces of functions $v = f(x)$, defined in some domain $D \in \mathbb{C}^n$. Although it doesn't have to be the case in general, we will think of the Hilbert space $L_2[a, b]$ of (complex) square integrable functions on a finite or infinite interval. with the inner product and norm

$$(f, g) = \frac{1}{b-a}\int_a^b dx f g^*, \qquad |f| = \left(\frac{1}{b-a}\int_a^b dx f f^*\right)^{1/2}.$$

Generalisation to more spatial dimensions is straightforward.

3.1 Integral operators

We define the linear operator with the *integration kernel* $K(x, y)$ as

$$h(x) = Kf(x) = \int_a^b K(x, y) f(y) dy.$$

Integration is a linear operation and so we have

$$\int_a^b K(x, y)\left(cf(y) + dg(y)\right) dy = c\int_a^b K(x, y) f(y) dy + d\int_a^b K(x, y) g(y) dy.$$

We define the unique adjoint operator K^* or $K^\dagger$

$$K^* f(x) = \int_a^b K^*(x,y) f(y) dy \quad \rightarrow \quad (Kf, g) = (f, K^* g).$$

An integral operator is symmetric or Hermitian if $K^* = K$. If it is real, this simply means for the kernel $K(x,y) = K(y,x)$, if it is complex, we have instead $K(x,y) = K(y,x)^*$. An integral operator can also have eigenvalues and eigenvectors. The eigenvectors of a Hermitian operator with distinct eigenvalues are orthogonal.

A kernel is said to be *symmetrically separable* if for some functions $h_i(x)$, we can write

$$K(x,y) = \sum_{ij} c_{ij} h_i(x) h_j(y).$$

The corresponding integral operator has a set of orthonormal eigenvectors $f_i(x)$ with non-zero eigenvalues λ_i. If $g(x)$ is orthogonal to all f_i, $Kg(x) = 0$. Each eigenspace is finite dimensional. If K has no other eigenvectors with non-zero eigenvalues, then

$$K(x,y) = \sum_j \lambda_j f_j(x) f_j(y),$$

which is exactly the analogue of having diagonalised the corresponding matrix, by expanding on the eigenbasis.

3.2 Differential operators

Differentiation is a linear operation, since

$$\frac{d}{dx}(af(x) + bg(x)) = a\frac{d}{dx}f(x) + b\frac{d}{dx}g(x).$$

So we can define a linear operator of the type

$$L = \left(f_{n+m}(x) \frac{d^{n+m}}{dx_i^n dx_j^m} \right)$$

where x_i are the various spatial dimensions and time, and we allow for partial derivatives mixed derivatives etc. In practice, we will concentrate on up to 3+1 space-time dimensions and up to second derivatives.

We are interested in differential equations of the type

$$Lu(x) = f(x) \quad \rightarrow \quad u(x) = L^{-1} f(x), \qquad L^{-1} L = 1.$$

and wish to find the Green's function L^{-1} to a given differential operator L.

Ex.: Invert d^2/dx^2. We have

$$\frac{d^2}{dx^2} G(x,t) = \delta(x,t).$$

We use the Heaviside function $H'(x) = \delta(x)$ to perform one integration

$$\frac{d}{dx} G(x,t) = H(x-t) + \alpha(t),$$

with $\alpha(t)$ some function. Integrating again, we have

$$G(x,t) = (x-t)H(x-t)dx + x\alpha(t) + \beta(t).$$

Now we want to solve the equation

$$\frac{d^2}{dx^2}u(x) = f(x),$$

with Dirichlet boundary conditions $u(0) = u(1) = 0$, and we have

$$u(x) = \int G(x,t)f(t)dt = \int_{-\infty}^{x} (x-t)f(t)dt + x\int_{-\infty}^{\infty} \alpha(t)f(t)dt + \int_{-\infty}^{\infty} \beta(t)f(t)dt.$$

Boundary conditions impose

$$\begin{aligned}
0 &= -\int_{-\infty}^{0} tf(t)dt + \int_{-\infty}^{\infty} \beta(t)f(t)dt, \\
0 &= \int_{-\infty}^{1} (1-t)f(t)dt + \int_{-\infty}^{\infty} \alpha(t)f(t)dt + \int_{-\infty}^{\infty} \beta(t)f(t)dt, \\
\to \beta(t) = tH(-t), \qquad &\to \alpha(t) = -1 + tH(t), \quad t \in]-\infty, 1], \quad 0, \quad t \in [1, \infty[.
\end{aligned}$$

so that finally

$$G(x,t) = (x-t)H(x-t) - x(1-t).$$

We see that the kernel satisfies the boundary conditions in the variable x. This is a general result.