Mathematical Methods

Lecture 11.
1/12-2008

1 Adjoint and self-adjoint operators

We again define the adjoint operator L^* or $L^\dagger$ through

$$(Lf, g) = (f, L^*g).$$

We should think in the distribution-sense, so that L^* is the operator that does something to functions v, in such a way that the inner product is equal to the inner product on the left hand side. In quantum mechanics, L^* is often written as $L^\dagger$, as the hermitian conjugate.

Ex.: Consider $L = d/dx$, $v(0) = 2v(1)$, for functions on $[0; 1]$. This defines a *domain* M for the operator as a certain set of function with certain boundary conditions. Then to get the adjoint we calculate

$$(u, Lv) = \int_0^1 u(x)\frac{dv}{dx}(x)dx = v(1)[u(1) - 2u(0)] - \int_0^1 v(x)\frac{du}{dx}dx = (L^*u, v).$$

This defines the adjoint to be $L^* = -d/dx$ if we also impose the condition $u(1) = 2u(0)$. Only when specifying the boundary condition is the differential operator completely determined. And these conditions determine the domain M^* for L^*, which may or may not be the same as M.

If $L = L^*$ it is *formally self-adjoint*. If also the boundary conditions are the same ($M^* = M$), if is *self-adjoint*.

Ex.: Consider $L = e^x d^2/dx^2 + e^x d/dx$ and $u'(0) = u(1) = 0$. We find the adjoint

$$(v, Lu) = \int_0^1 v\left(e^x u'\right)' dx = u'(1)v(1)e^2 + v'(0)u(0) + \int_0^1 u(e^x v'' + e^x v')dx.$$

So L is formally self-adjoint, and in fact self-adjoint, since by choosing $v'(0) = v(1) = 0$, the boundary terms disappear.

2 Distributions and differentation

We briefly touch on the a generalisation of differentiation to distributions. Consider a distribution $f(x)$, and its derivative $f'(x)$. We define it through

$$\int_{-\infty}^{\infty} \phi(x)f'(x)dx = -\int_{-\infty}^{\infty} \phi'(x)f(x)dx,$$

where $\phi(x)$ is a test function, so that the boundary terms vanish.

Ex.: Consider $f(x)\delta(x - x_0)$. What is $f'(x)$?

$$\int_{-\infty}^{\infty} \phi(x)\delta'(x - x_0)dx = -\int_{-\infty}^{\infty} \phi'(x)\delta(x - x_0)dx = \phi'(x_0).$$

So whereas the Dirac delta function evaluates a test function at x_0, the derivatives of the delta function evuates the derivative at x_0, with a minus sign.

3 Boundary value problems and Greens functions

Ex.: Consider the differential equation

$$-\frac{d^2}{dx^2}u(x) = -f(x), \qquad u(0) = a, \quad u(1) = b.$$

The boundary conditions are Dirichlet, but not trivial (or homogeneous). So the domain is not as we had before functions that vanish at the boundary, for which we know that Greens function. Let us write

$$(Lv, w) = (v, Lw), \qquad L = -\frac{d^2}{dx^2},$$

with v having homogeneous boundary conditions, but w not. L is self-adjoint, so the definition makes sense. Let us *define* Lv by this equation, and we have

$$(Lv, w) = -\int_0^1 vw''dx - v'(1)w(1) + v'(0)w(0),$$

where we have used the boundary conditions of v. Rather than interpreting this as an operator bit plus some boundary bits, let us rewrite

$$\int_0^1 v\left(-w'' + w(1)\delta'(x-1) - w(0)\delta'(x)\right)dx,$$

and define Lw to be the thing in the bracket. If w obeys the boundary conditions of v, the boundary bit is zero. Otherwise, L acting on a general boundary condition function w is a sum of terms involving the original L and some delta functions at the boundaries.

We now instead have to solve

$$Lu = -f(x) + b\delta'(x-1) - a\delta(x),$$

where $f(x)$ is the source away from the boundary, and the two delta-functions encode the boundary conditions. Let us split up like $u = u_1 + u_2$, and we have

$$\begin{aligned} Lu_1(x) &= -f(x), \\ Lu_2(x) &= b\delta'(x-1) - a\delta(x). \end{aligned}$$

Since we have that

$$\begin{aligned} &LG(x,y) = -\delta(x-y), \quad G(x,y) = (x-y)H(x-y) - x(1-y), \\ &L\frac{dG(x,y)}{dy} = \delta'(x-y), \end{aligned}$$

we have that

$$u_1(x) = \int_{-\infty}^{\infty} G(x,y)f(y)dy, \qquad u_2(x) = b\left(\frac{dG(x,y)}{dy}\right)_{y=1} - a\left(\frac{dG(x,y)}{dy}\right)_{y=0}$$

and so

$$u = u_1 + u_2 = \int_{-\infty}^{\infty}(x-y)H(x-y)f(y)dy - \int_{-\infty}^{\infty}x(1-y)f(y)dy + a(1-x) + bx.$$

Consider the most general self-adjoint second order differential operator

$$Lu = -\frac{1}{w(x)}(p(x)u')' + q(x)u.$$

Using similar methods as above, we find (for $q(x) = 0$ for simplicity)

$$G(x,y) = -H(x-y)\int_y^x \frac{du}{p(u)} + \alpha(y)\int_0^x \frac{du}{p(u)} + \beta(y).$$

The derivative wrt x is

$$\frac{d}{dx}G(x,y) = -\delta(x-y)\int_y^x \frac{du}{p(u)} - H(x-y)\frac{1}{p(x)} + \alpha(y)\frac{1}{p(x)}.$$

Because of the step-function, this derivative has a jump at $x = y$ with height $-1/p(x = y)$. This suggests another method for finding the Greens function, by constructing it using the following requirements:

- $G(x,y)$ with y fixed is a continuous function of x.
- The derivative $\partial_x G(x,y)$ with y fixed is a discontinuous function of x, with a jump $-1/p(x)$.

Ex.: Consider $L = -d^2/dx^2$, $u(0) = u'(0) = 0$. We have

$$\frac{d^2}{dx^2}G = -\delta(x-y), \quad G(0,y) = \frac{d}{dx}G(0,y) = 0.$$

Away from $x = y$, we have the solution

$$G(x,y) = \alpha(y)x + \beta(y).$$

For $x < y$, and in order to obey the boundary conditions, we have $\alpha = \beta = 0$. Then at $x = y$ the derivative must change to -1 but the function must still be continuous, and we have

$$G(x,y) = 0, \;\; < y \qquad G(x,y) = -x + y, \;\; x > y \quad \to \quad G(x,y) = -(x-y)H(x-y).$$

Ex.: Then the solution to

$$-\frac{d^2}{dx^2}u = f(x), \qquad u(0) = u'(0) = 0$$

is

$$u(x) = \int_0^1 (x-y)H(x-y)f(y)dy = \int_0^x (x-y)f(y)dy.$$

Ex.: Now the same thing but with non-homogeneous boundary conditions,

$$-\frac{d^2}{dx^2}u = f(x), \qquad u(0) = b, \quad u'(0) = a.$$

We take a lucky guess and consider

$$L^* = -\frac{d^2}{dx^2}u, \quad v(1) = v'(1) = 0,$$

and find

$$(Lu, v) = (u, L^* v) \quad \rightarrow \quad Lu = f(x) - a\delta(x) - b\delta'(x).$$

Since the Greens function for the homogeneous boundary condition problem is

$$G(x, y) = -(x - y)H(x - y), \qquad \delta(x) = LG_{y=0}, \qquad \delta'(x) = L\left(\frac{dG}{dy}\right)_{y=0} = -x,$$

we have

$$u_1(x) = \int_0^x (x - y)f(y)dy, \qquad u_2(x) = ax + b, \quad \rightarrow \quad u(x) = u_1(x) + u_2(x).$$

Ex.: Yet another type of (mixed) boundary conditions,

$$-\frac{d^2}{dx^2}u = f(x), \qquad u(0) = 0, \quad u(1) = 0.$$

The Greens function has to obey the boundary conditions as a function of the x variable. First we guess that

$$G(x, y) \sim x, \quad x < y, \quad G(x, y) \sim 1 - x, \quad x > y.$$

But this is not continuous at $x = y$. So now use a trick; multiply the "one end solution" by the "other end solution evaluated at $x = y$", to get

$$G(x, y) = x(1 - y), \quad x < y, \quad G(x, y) = (1 - x)y, \quad x > y.$$

This obeys all the requirements including the jump in the derivative at $x = y$. Although not unique, this Greens function has the extra feature that it is symmetric in x and y.

Then to solve the problem, we have

$$u(x) = \int_0^1 G(x, y)f(y)dy = (1 - x)\int_0^x yf(y)dy + x\int_x^1 (1 - y)f(y)dy.$$

Ex.: The same boundary conditions, but now inhomogeneous, and a more complicated differential operator

$$\left(\frac{d^2}{dx^2} + k^2\right)u = f(x), \qquad u(0) = a, \quad u(1) = b.$$

First, find the Greens function with homogeneous boundary conditions by construction. General solutions are $\sin kx$, $\cos kx$, and to satisfy the boundary conditions,

$$G(x, y) \sim \sin kx, \quad x < y, \qquad G(x, y) \sim \cos k(1 - x), \quad x > y.$$

But this doesn't work, so now cross-multiply as above,

$$G(x, y) \propto \sin kx \cos k(1 - y), \quad x < y, \qquad G(x, y) \propto \sin ky \cos k(1 - x), \quad x > y.$$

But differentiating wrt x, we find that the jump is $-k\cos k$ instead of -1. So we normalise to finally get

$$G(x, y) = \frac{\sin kx \cos k(1 - y)}{k \cos k}, \quad x < y, \qquad G(x, y) = \frac{\sin ky \cos k(1 - x)}{k \cos k}, \quad x > y.$$

Writing as before $u = u_1 + u_2$, we have

$$Lu_1(x) = -f(x), \qquad Lu_2(x) = b\delta(x-1) - a\delta'(x) = bLG(x,0) - aL\frac{d}{dy}G(x,0),$$

so that

$$u(x) = \int G(x,y)f(y)dy + b\frac{\sin kx}{k\cos k} + a\frac{\cos k(1-x)}{k\cos k}.$$

Ex.: All of this generalizes to the self-adjoint operator

$$L = -\frac{d}{dx}\left(p(x)\frac{d}{dx}\right) + q(x),$$

with general boundary conditions,

$$B_1 = a_1u(0) + b_1u(1) + c_1u'(0) + d_1u'(1) = \alpha,$$
$$B_2 = a_2u(0) + b_2u(1) + c_2u'(0) + d_2u'(1) = \beta,$$