Mathematical Methods

Lecture 2.
22/9-2008

1 Differentiation, continued

Theorem: If f is differentiable in a domain D and $f'(z) = 0$ everywhere in D, f is a constant. If Ref, Imf or $|f|$ is constant, f is constant.

Theorem: Within the radius of convergence, $|z - z_0| < R$ (where the series is absolutely convergent), we can take the derivative of a power series term by term,

$$\left(\sum a_n (z - z_0)^n\right)' = \sum n a_n (z - z_0)^{n-1},$$

and the result is convergent. This applies also to any higher order derivative. Hence

$$a_n = \frac{f^{(n)}(z_0)}{n!}.$$

where $f^{(n)}$ denotes differentiation n times. This shows that the power series is in fact the Taylor series,

$$f(z) = \sum \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n.$$

Turns out *all* complex differentiable functions can be expressed as power series, and infinitely differentiable and integrable. More about this later.

Ex. $f(z) = \frac{1}{1-z} = \sum_n z^n$. The series has radius of convergence $R = 1$. Then

$$f'(z) = \sum_n n z^{n-1} = \frac{1}{(1-z)^2}.$$

Taylor expansion around $z_0 = 0$,

$$f(0) = 1, \quad f'(0) = 1, \quad f''(0) = 2, \quad f^{(3)}(0) = 6, \quad f^{(4)}(0) = 24, \quad ... \quad f^{(n)}(0) = n!.$$

$$f(z) = \sum_n \frac{f^{(n)}}{n!}(z - z_0)^n = \sum_n z^n.$$

Around $z_0 = 2$?

$$f^{(n)} = n!(-1)^{n+1}, \qquad f(z) = \sum_n \frac{f^{(n)}}{n!}(z - z_0)^n = \sum_n (-1)^{n+1}(z-2)^n, \qquad R = \lim_{n\to\infty} 1 = 1.$$

Around $z_0 = -1$?

$$f^{(n)} = \frac{n!}{2^{n+1}}, \qquad f(z) = \sum_n \frac{f^{(n)}}{n!}(z - z_0)^n = \sum_n \frac{1}{2^{n+1}}(z+1)^n, \qquad R = \lim_{n\to\infty} 2 = 2.$$

What does it mean?
The definition of differentiability restricts functions to a fairly small subset, represented by the Cauchy-Riemann equations. On the other hand, once a function is known to be complex differentiable, we also know that it has a power series expansion in some region, is infinitely many times differentiable and that the real and imaginary parts are harmonic funtions. Differentiation can for instance be done term by term in the power series, which turns out to be the Taylor series.

2 Integration

Consider a path $s(t)$, $t \in [a,b]$, running from $s(a) = z_0$ to $s(b) = z_1$ in the complex plane. It is a *smooth path* if the mapping $s : [a,b] \to \mathbb{C}$ is differentiable and the derivative is continuous. Along such a smooth path, we define the contour integral of the complex function $f : D \to \mathbb{C}$ on the domain D to be

$$\int_s f = \int_a^b f(s(t))s'(t)dt,$$

Ex.

$$f(z) = z^2, \qquad s(t) = t^2 + it, \qquad t \in [0,1].$$

$$\int_s f = \int_0^1 dt f(s)s'(t) = \int_0^1 dt(t^2+it)^2(2t+i) = -\frac{2}{3} + \frac{2}{3}i.$$

The length of the path $s(t)$ is

$$L(s) = \int_a^b |s'(t)|dt.$$

Ex.

$$s(t) = z_1(1-t) + z_2 t, \qquad t \in [0,1], \qquad L(s) = \int_0^1 dt|s'(t)| = \int_0^1 dt|z_2 - z_1| = |z_2 - z_1|.$$

$$s(t) = z_0 + re^{it}, \qquad t \in [0,2\pi], \quad r > 0, \qquad L(s) = \int_0^{2\pi} dt|re^{it}| = 2\pi r.$$

The integral has the usual features ($c \in \mathbb{C}$),

$$c\int_s (f_1 \pm f_2) = \int_s (cf_1) \pm \int_s (cf_2),$$

and we can glue together paths into a piecewise smooth path

$$\int_{s_1+s_2} f = \int_{s_1} f + \int_{s_2} f,$$

under the assumption that $s_1 + s_2$ is continuous. We also note that

$$\int_{-s} f = -\int_s f.$$

Ex. A very useful closed contour is

$$s_1(t) = t, \quad z \in [\epsilon, R], \qquad s_2(t) = R(\cos(t) + i\sin(t)), \quad z \in [0, \pi],$$
$$s_3(t) = t, \quad z \in [R, \epsilon], \qquad s_4(t) = \epsilon(-\cos(t) + i\sin(t)), \quad z \in [0, \pi].$$

Theorem If $f : D \to \mathbb{C}$ is continuous, $F : D \to \mathbb{C}$ obeys $F' = f$ and s is a contour in D from z_0 to z_1, then

$$\int_s f = F(z_1) - F(z_0).$$

Ex. Along any contour $s(t)$ from $z_0 = 0$ to $z_1 = 1 + i$

$$f(z) = z^2, \qquad \int_s z^2 dz = \frac{1}{3}z_1^3 - \frac{1}{3}z_0^3 = -\frac{2}{3} + \frac{2}{3}i.$$

Theorem Power series $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ can be integrated within their radius of convergence,

$$F(z) = \sum_{n=0}^{\infty} \frac{a_n}{n+1}(z - z_0)^{n+1}, \qquad F' = f,$$

and the resulting series converges. The contour integral does not depend on the contour, and so we have

Theorem If $f : D \to \mathbb{C}$ is continuous, then the statements a) f has an antiderivative F in D; b) $\int_s f = 0$ for every closed contour s in D; and c) $\int_s f$ depends only on the endpoints of s for any contour in D; are equivalent.

Ex. $f(z) = |z|$ is not differentiable, so different contours give different integrals.

$$s(t) = it, \quad t \in [0, 1], \qquad \int_s |z| dz = \frac{i}{2},$$

$$s(t) = s_1(t) + s_2(t), \quad s_1(t) = t, \quad t \in [0, 1], \quad s_2(t) = e^{it}, \quad t \in [0, \pi/2] \qquad \int_s |z| dz = i - \frac{1}{2},$$

What does it mean?

Differentiation has an opposite operation, integration. We need a contour to integrate along, although if there exists an antiderivative, the choice of contour doesn't matter. Within its radius of convergence, a power series does have an antiderivative, found by integratng term by term.

3 Standard functions

The Exponential function is defined by its power series

$$\exp(z) = \sum_n \frac{z^n}{n!} = e^z,$$

which is absolutely convergent for any $z \in \mathbb{C}$. We have the relations

$$(\exp(z))' = \exp(z), \qquad \exp(z_1 + z_2) = \exp(z_1)\exp(z_2).$$

A function is said to be periodic with period T if $f(z+T) = f(z)$. $\exp(z)$ only has the periods $T = 2n\pi i$, $n \in \mathbb{Z}$.

The Trigonometric functions are defined by

$$\cos(z) = \sum_n (-1)^n \frac{z^{2n}}{(2n)!}, \qquad \sin(z) = \sum_n (-1)^n \frac{z^{2n+1}}{(2n+1)!},$$

They are also absolutely convergent for any $z \in \mathbb{C}$. Also, we have

$$(\cos(z))' = -\sin(z), \qquad (\sin(z))' = -\cos(z), \qquad \cos(z) = \frac{e^{iz} + e^{-iz}}{2}, \qquad \sin(z) = \frac{e^{iz} - e^{-iz}}{2i}.$$

$\sin(z) = 0 \leftrightarrow z = n\pi$, $\cos(z) = 0 \leftrightarrow z = (n+1/2)\pi$. Hence we can define

$$\tan(z) = \frac{\sin(z)}{\cos(z)}, \quad z \neq (n+1/2)\pi, \qquad (\tan(z))' = 1 + \tan^2(z),$$

and similarly for the other trigonometric functions.

The Hyperbolic functions are defined by

$$\cosh(z) = \frac{e^z + e^{-z}}{2}, \qquad \sinh(z) = \frac{e^z - e^{-z}}{2}, \qquad (\cosh(z))' = \sinh(z), \qquad (\sinh(z))' = \cosh(z)$$

Hence, the fundamental function is $\exp(z)$, and the rest follows.

The Logarithm functions is the inverse of $\exp(z)$. However, since $\exp(z)$ is not one-to-one, some care is required. We wish to define

$$\log(z), \quad z \neq 0, \quad \text{from} \quad w = \log(z) \leftrightarrow \quad z = e^w.$$

We write

$$z = re^{i\theta}, \quad w = u + iv \to r = e^u, \quad u = \log(r).$$

but

$$e^{i\theta} = e^{iv} \to v = \theta + tn\pi, \quad n \in \mathbb{Z} \to \log(z) = \log(|z|) + i(\arg(z) + 2n\pi).$$

and so log is a *multivalued function*. The *principal* value is defined by

$$\mathrm{Log}(z) = \log(|z|) + i\arg(z), \quad \arg(z) \in]-\pi, \pi]$$

In the *cut plane* $\mathbb{C}_\pi = \mathbb{C}\backslash\{x \leq 0\}$, $\mathrm{Log}(z)$ is continuous and differentiable

$$(\mathrm{Log}(z))' = 1/z.$$

Note that although the principal logarithm Log is defined for all $\mathbb{C}$, it is just not continuous or differentiable along the negative real half-axis.

We define a general exponential function to be

$$z^a = \exp(a\mathrm{Log}(z)).$$

What does it mean?

The exponential function is the basis of all trigonometric and hyperbolic functions. We choose to define these functions as power series, since these are convergent in all of $\mathbb{C}$. The inverse of the exponential function, the logarithm, suffers from multivalued-ness in terms of its argument. This leads to the introduction of principal values and cuts. More about this later.

4 Winding number and Cauchy's theorem

Theorem: Consider a path $s : [a, b]$ not going through $z = 0$. The exists a continuous choice of argument for s. All choices differ by $2n\pi$.

Ex.

$$s(t) = re^{4\pi it}, \quad t \in [0, 1], \qquad \theta(s) = 4\pi t + 2n\pi$$

We define the *winding number* of a path $s : [a, b]$ around the point z_0 as

$$n_w(s, z_0) = \frac{\theta(b) - \theta(a)}{2\pi}.$$

where $\theta(z)$ is such a continuous choice of argument. For closed paths the winding number is an integer. For a general path it is not. Combining paths amounts to adding up the winding number $n_w(s_1 + s_2, z_0) = n_w(s_1, z_0) + n_w(s_2, z_0)$. Note that the winding number comes with a sign, which is **positive in the anti-clockwise direction** and **negative in the clockwise direction**.

We also have for a closed path s

$$n_w(s, z_0) = \frac{1}{2\pi i} \int_s \frac{1}{z - z_0} dz.$$

For an open path, we generalise this to

$$n_w(s, z_0) = \frac{1}{2\pi} \mathrm{Im} \int_s \frac{1}{z - z_0} dz$$

Cauchy's Theorem: Let f be differentiable in a domain $D \subset \mathbb{C}$, and s a closed contour in D, which has $n_w(s, z) = 0$, for all $z \notin D$. Then $\int_s f = 0$.

Theorem: If f be differentiable in a domain D and s a closed contour with all its inside $I(s) = \{z \in \mathbb{C} | n_w(s, z) = 0\}$ in D, then $\int_s f = 0$.

Theorem: Given D, $\int_s f = 0$ for all closed contours s in D and all differentiable functions f in D if and only if D is simply connected.

Simply connected means that all closed contours s have $n_w(s, z) = 0$ for all $z \notin D$ ("D has no holes").

What does it mean?

The Cauchy theorem is central to complex analysis. It says that the differentiable functions on a simply connected domain have antiderivatives and so integrals along closed contours are zero. In particular, we can move integration paths in any domain around as it suits us, as long as we don't "cross any holes".

5 Taylor and Laurent series

Cauchy's integral formula: For f differentiable in the disc $N_R(z_0)$ of radius R around z_0, and a path $s(t) = z_0 + re^{it}, \quad 0 < r < R, \quad t \in [0, 2\pi]$, then for any point w, $|w - z_0| < r$, we have

$$f(w) = \frac{1}{2\pi i} \int_s \frac{f(z)}{z - w} dz.$$

5.1 Taylor series

Theorem: For f differentiable in the disc $N_R(z_0)$ of radius R around z_0, and a path $s(t) = z_0 + re^{it}, \quad 0 < r < R, \quad t \in [0, 2\pi]$, then

$$f(z_0 + h) = \sum_{n=0}^{\infty} a_n h^n,$$

with

$$a_n = \frac{1}{2\pi i} \int_s \frac{f(z)}{(z - z_0)^{n+1}} dz.$$

The series is absolutely convergent for $|h| < R$.

Taylor series: If f is differentiable in D, then all higher derivatives exist in D and in any disc $N_R(z_0)$ the power series expansion is the Taylor expansion

$$f(z_0 + h) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} h^n$$

is convergent with, for $s(t) = z_0 + re^{it}, \quad 0 < r < R, \quad t \in [0, 2\pi]$,

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_s \frac{f(z)}{(z - z_0)^{n+1}} dz.$$

Morera's Theorem: If f is continuous in a domain D and $\int_s f = 0$ for all closed s, then f is differentiable.

Note that real continuous functions always have antiderivatives. Only some complex continuous functions are differentiable, but all of these have antiderivatives in simply connected (sub-)domains like discs.

Cauchy's estimate: I f is differentiable for $|z - z_0| < R$, and $|f(z)| \leq M$ on the circle $|z - z_0| = r < R$, then

$$|f^{(n)}(z_0)| \leq \frac{Mn!}{r^n}, \qquad \forall n \geq 0.$$

Liouville's Theorem: If f is differentiable and bounded in $\mathbb{C}$, then it is a constant.

Fundamental theorem of algebra: Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_0$, with $n \geq 0$ and $a_i \in \mathbb{C}$. Then there exists $w \in \mathbb{C}$ so that $P(w) = 0$.

What does it mean?

Cauchy's integral formula shows that the value of a differentiable function is completely determined by the value on a circle around it (or indeed any closed contour). More importantly, we see that all differentiable functions are infinitely differentiable and have a Taylor expansion in a region around any point in the domain. This leads to a number of interesting theorems, highlighting the strength of differentiability.