Mathematical Methods

Lecture 3.
29/9-2008

1 Singularities

1.1 Zeros

A *zero* z_0 of $f : D \to \mathbb{C}$ is where $f(z_0) = 0$. With

$$f(z) = \sum_n a_n (z - z_0)^n,$$

a zero has *finite order* m if $a_m \neq 0$ but $a_i = 0$, $i < m$. A zero is *isolated* is it has a neighbourhood with no other zeros. A zero of finite order is isolated.

Theorem: If f is differentiable in D and there exists a set of zeros with a limit point z_0 in D, then f is zero in all of D.

Identity theorem: If f and g are differentiable and $f(z) = g(z)$ on a set with a limit point in the domain D, the $f = g$ throughout D.

$g : D \to \mathbb{C}$ is an *extension function* of $f : S \subset D \to \mathbb{C}$ if $f = g$ in S.

Ex.

$$f(z) = \frac{1}{1-z}, \quad z \neq 1$$

is an extension function of

$$f(z) = \sum_n z^n, \quad |z| < 1$$

Ex. $f(z) = \sin(z)$, $z \in \mathbb{C}$ is the only differentiable extension of $f(x) = \sin(x)$, $x \in \mathbb{R}$.

1.2 Extrema

Because complex numbers are not ordered, we cannot speak of $f(z_1) < f(z_2)$. We can however consider the modulus $|f(z)| > 0$ of the function. A local maximum (minimum) of a complex function is a point z_0 so that $|f(z)| \leq (\geq) |f(z_0)|$ when $|z - z_0| < \epsilon$ for some ϵ.

Theorem: If a differentiable function has a maximum in a domain, the function is constant in that domain. If a non-zero differentiable function has a minimum in a domain, the function is constant in that domain.

Theorem: If a differentiable function is not constant, the maximum of its modulus occurs on the boundary of the set. If a differentiable function is not constant, the minimum of its modulus occurs on the boundary or where the function is zero.

Ex. $f(z) = z^2$ on $|z| \leq 1$. $|f(z)| = x^2 + y^2$, maximum on boundary $|z| = 1$ and minimum at $z = 0$.

1.3 Laurent series and isolated singularities

We can generalise the concept of a power series to include negative powers and represent a function by

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n.$$

A normal ($n \geq$) power series converges *inside* some radius R_2, and a $n < 0$ power series converges *outside* some other radius R_1. In case $R_2 > R_1$, there is an *annulus*, where both converge.

Laurent's Theorem: If f is differentiable in an annulus $0 \leq R_1 \leq |z - z_0| \leq R_2 \leq \infty$, there exists a *Laurent series*

$$f(z_0 + h) = \sum_{n=0}^{\infty} a_n h^n + \sum_{n=1}^{\infty} b_n h^{-n},$$

with $\sum_{n=0}^{\infty} a_n h^n$ convergent for $|h| < R_2$ and $\sum_{n=1}^{\infty} b_n h^{-n}$ converges for $|h| > R_1$. Using the contour $s(t) = z_0 + re^{it}$, $t \in [0, 2\pi]$, $R_1 < r < R_2$, we have

$$a_n = \frac{1}{2\pi i} \int_s \frac{f(z)}{(z - z_0)^{n+1}} dz, \quad b_n = \frac{1}{2\pi i} \int_s f(z)(z - z_0)^{n-1} dz.$$

In practice, we will not use these integrals to calculate whole Laurent series. Note that the coefficients are not the derivatives as for the Taylor series.

Ex. For $z \neq 0$,

$$f(z) = e^z + e^{1/z} = \sum_{n=0}^{\infty} \frac{z^n}{n!} + \sum_{n=0}^{\infty} \frac{z^{-n}}{n!}, \qquad c_m = 1/m!, \quad c_{-m} = 1/m!, \quad c_0 = 2.$$

Ex. For $0 < |z| < 1$,

$$f(z) = \frac{1}{z} + \frac{1}{1 - z} = \sum_{n=-1}^{\infty} c_n z^n, \qquad c_n = 1.$$

Ex. For $1 < |z| < 2$,

$$f(z) = \frac{1}{z - 1} - \frac{1}{z - 2} = \frac{1}{z(1 - 1/z)} + \frac{1}{2(1 - z/2)} = \sum_{n=-\infty}^{\infty} c_n z^n, \qquad c_{n<0} = 1, \quad c_{n\geq 0} = 2^{-m-1}.$$

The Laurent expansion allows us to categorize singularities. If f is differentiable everywhere but at some point z_0, so for the annulus $0 < |z - z_0| < R$ for some R, we say that z_0 is an *isolated singularity*. We can write down the Laurent series giving the coefficients b_n (of the negative power terms). If

all $b_n = 0$, we have a *removable singularity*, and we can set $f(z_0) = a_0$. If

a finite number, m, of the $b_n \neq 0$, we have a *pole of order m* at z_0. If

infinitely many $b_n \neq 0$, we have an *isolated essential singularity*.

Ex. $f(z) = \sin(z)/z$, $z \neq 0$ has the Laurent expansion $f(z - z_0) = 1 - z^2/3! + z^4/5! - ....$ With $f(z_0) = 1$ f is differentiable in $\mathbb{C}$.

Ex. $f(z) = \sin(z)/z^4$, $z \neq 0$ has the Laurent expansion $f(z - z_0) = 1/z^3 - z/3! + z/5! - ....$ It has a 3rd order (*triple*) pole at $z_0 = 0$.

Ex. $f(z) = \sin(1/z)$, $z \neq 0$ has the Laurent expansion $f(z - z_0) = 1/z - 1/z^3 3! + 1/z^5 5! - ....$ It has an isolated essential singularity at $z_0 = 0$.

Theorem: For f differentiable in $0 < |z - z_0| < R$, the statements a)z_0 is a removable singularity; b) $\lim_{z\to z_0} f(z) < \infty$; c) f is bounded in a neighbourhood of z_0; are equivalent.

Therefore if z_0 is not a removable singularity, $b_n \neq 0$ for some n, f is unbounded in any neighbourhood of z_0 ("goes to infinity at the singularity").

Theorem: For f differentiable in $0 < |z - z_0| < R$ it has a pole of order m if and only if

$$\lim_{z\to z_0} (z - z_0)^m f(z) = l \neq 0, \quad < \infty,$$

since in that case the limiting operation picks up the b_m coefficient $= l$.

Theorem: f has a pole of order m at z_0 if and only if $1/f$ has a zero of order m there.

Theorem: If f has pole at z_0, then $\lim_{z\to z_0} |f(z)| = \infty$ (modulus is singular at a singularity).

Weierstrass-Casorati-Poincare Theorem: In every neighbourhood of an essential singularity, f takes on all values in $\mathbb{C}$ except at most one(!)

What is going on? Consider the integral of a Laurent series around the closed loop $s(t) = z_0 + re^{it}$, $t \in [0, 2\pi]$, in the case of an isolated singularity at z_0. Compute first for all integers n

$$\int_s \frac{1}{(z - z_0)^n} dz = \int_0^{2\pi} \frac{ire^{it}}{r^n e^{int}} dt = \frac{i}{r^{n-1}} \int_0^{2\pi} e^{i(1-n)t} dt = 0, \quad n \neq 1.$$

If $n = 1$ we instead have

$$\int_s \frac{1}{(z - z_0)^1} dz = n_w(s, z_0) = 1.$$

For positive powers

$$\int_s (z - z_0)^n dz = 0$$

because the function is differentiable. Therefore,

$$\int_s f(z) dz = \int_s \frac{b_1}{z - z_0} dz + 0 = 2\pi i n_w(s, z_0) b_1 = 2\pi i b_1.$$

In order to extract the other coefficients, we need to multiply by the appropriate power of $(z - z_0)$,

$$\int_s f(z)(z - z_0)^n dz = \int_s \frac{b_{n+1}}{z - z_0} dz + 0 = 2\pi i b_{n+1}.$$

And

$$\int_s f(z)(z - z_0)^{-n} dz = \int_s \frac{a_{n-1}}{z - z_0} dz + 0 = 2\pi i a_{n-1}.$$

1.4 Infinity

We can extend $\mathbb{C}$ with the point ∞ by mapping the complex plane onto a sphere. The point ∞ is then the North Pole (not to be confused with a singularity-type pole). We can think of $\{z : |z| > R\}$ as a neighbourhood around ∞ and define continuity and differentiability. For $f(z)$ differentiable for $|z| > R$, we say that f has a removable or isolated essential singularity or pole of order m at ∞, if $g(z) = f(1/z)$, $0 < |z| < 1/R$, has such a singularity at $z = 0$. Similarly, we say that f has a zero of order m at ∞ if g has it at $z = 0$.

Ex. $f(z) = 1/z$, $|z| > 0$. $g(z) = z$, $|z| > 0$ has a removable singularity at $z = 0$, and so f has a removable singularity at ∞.

Ex. $f(z) = z$. $g(z) = 1/z$, $|z| > 0$ has single pole at $z = 0$, and so f has a single pole at ∞.

Ex. $f(z) = e^z$. $g(z) = e^{1/z}$, $|z| > 0$ has an isolated essential singularity at $z = 0$, and so f has an isolated essential singularity at ∞.

Ex. $f(z) = 1/\sin(z)$, $z \neq n\pi$, $n \in \mathbb{Z}$. $g(z) = 1/\sin(1/z))$. Doesn't work, since f is not differentiable for $|z| > R$ for any R. There are infinitely many isolated singularities of f as $z \to \infty$ (or of g as $z \to 0$). We say that f has an *essential singularity* at ∞.

A function is said to be *meromorphic* in a domain D if it is differentiable everywhere in D except at poles. Meromorphic functions on the extended complex plane can always be written as *rational functions* (a fraction of two polynomials). This is not the case in the un-extended complex plane.

What does it mean: We have finally extended the boring differentiable functions to meromorphic functions, a much broader class. Poles are singularities where the function "goes to infinity" as a particular power $(z - z_0)^m$, encoded in the coefficient of the leading order term (the one with the largest negative power) in the corresponding Laurent series. It is no longer a Taylor series and we cannot extract the coefficients by simple differentiation.

2 Residues

We define the *residue* at a point z_0 in terms of the Laurent series,

$$f(z_0 + h) = \sum_{n=0}^{\infty} a_n h^n + \sum_{n=1}^{\infty} b_n h^{-n},$$

by (with $s(t) = z_0 + re^{it}$, $t \in [0, 2\pi]$)

$$\text{res}(f, z_0) = b_1 = \frac{1}{2\pi i} \int_s f(z) dz.$$

We define a *simple loop* to be a closed contour in a domain D for which all points either have winding number 0 or 1. We then have the crucial

Cauchy's Residue Theorem: Let s be a simple loop in a domain D and f differentiable everywhere except at a finite number of isolated singularities, z_r, $r = 1, 2, .., n$, all inside s. Then

$$\int_s f(z) dz = 2\pi i \sum_r \text{res}(f, z_r).$$