Mathematical Methods

Lecture 4.
6/10-2008

1 Residues

We define the *residue* at a point z_0 in terms of the Laurent series,

$$f(z_0+h)=\sum_{n=0}^{\infty}a_nh^n+\sum_{n=1}^{\infty}b_nh^{-n},$$

by (with $s(t)=z_0+re^{it}$, $t\in[0,2\pi]$)

$$\text{res}(f,z_0)=b_1=\frac{1}{2\pi i}\int_s f(z)dz.$$

We define a *simple loop* to be a closed contour in a domain D for which all points either have winding number 0 or 1. We then have the crucial

Cauchy's Residue Theorem: Let s be a simple loop in a domain D and f differentiable everywhere except at a finite number of isolated singularities, z_r, $r=1,2,..,n$, all inside s. Then

$$\int_s f(z)dz=2\pi i\sum_r \text{res}(f,z_r).$$

How to find the residue?:

Simple pole: If z_0 is a simple pole of $f(z)$, then

$$\text{res}(f,z_0)=\lim_{z\to z_0}(z-z_0)f(z).$$

Why? Because this singles out the $b_1/(z-z_0)$ term, and all the others go to zero in the limit (note that because there is a simple pole, there exists a Laurent series, with no negative powers except for $b_1/(z-z_0)$).

Simple pole for a fraction: If $f(z)=p(z)/q(z)$ has simple pole at z_0, $p(z_0)\neq 0$, $q(z_0)=0$, then

$$\text{res}(f,z_0)=\frac{p(z_0)}{q'(z_0)}.$$

Why? If the fraction has a single pole at z_0, p,q must both be differentiable at z_0. Hence they have Taylor series around z_0, and we can write

$$\lim_{z\to z_0}(z-z_0)f(z)=\lim_{z\to z_0}\frac{(z-z_0)(a_0+a_1(z-z_0)+...)}{c_0+c_1(z-z_0)+...}$$

Because $q(z_0)=0$, $c_0=0$, and the limit is $a_0/c_1=p(z_0)/q'(z_0)$.

Ex.

$$f(z)=\frac{\cos(\pi z)}{1-z^{243}}\to\text{res}(f,1)=\frac{\cos(\pi z)}{-243z^{243}}|_{z=1}=\frac{1}{243}.$$

General pole: If z_0 is a pole of order m of $f(z)$, then

$$\mathrm{res}(f, z_0) = \lim_{z \to z_0} \left(\frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} ((z - z_0)^m f(z)) \right).$$

Why? Again, we want to single out b_1, but now we have terms $b_i(z - z_0)^i$, $i = 1, .., m$. First we multiply by $(z - z_0)^m$ so that we only have positive powers. Then we differentiate all the $b_m, b_{m-1}, .., b_2$ terms away (that's $m - 1$ times). Finally we have to divide out the $(m - 1)!$ factor we got from differentiating. Then take the limit, and only b_1 is left. Note that we do not need to know the Laurent series, we just need to know that it exists.

Ex. Triple pole at $z = 1$

$$\left(\frac{z+1}{z-1}\right)^3 \to \mathrm{res}(f, 1) = \lim_{z \to 1} \frac{1}{2!} \left((z - 1)^2 f(z)\right)'' = 6.$$

Find the Laurent series: If one can guess the Laurent series, up to and including b_1, that's the residue.

Ex.

$$f(z) = \frac{1}{z^2 \sin(z)} = \frac{1}{z^2(z - z^3/3! + ...)} = \frac{1}{z^3}\left(1 + \frac{z^2}{6} + ...\right) = \frac{1}{z^3} + \frac{1}{6z} + ...$$

$$\mathrm{res}(f, 0) = 1/6.$$

2 Evaluation of real integrals

Trick: Real integrals over a period.

$$\int_0^{2\pi} F(\cos(t), \sin(t))dt$$

with F some function. Use the unit circle $s(t) = e^{it}$, $t \in [0, 2\pi]$. Then

$$\cos(t) = (z + 1/z)/2, \qquad \sin(t) = (z + 1/z)/2i.$$

hence

$$\int_0^{2\pi} F(\cos(t), \sin(t))dt = \int_s F((z + 1/z)/2, (z + 1/z)/2i)\frac{dz}{iz} = 2\pi i\Sigma,$$

where Σ is the sum of residues of $F(z)$ inside s. Note that normally we have an integral in z and put in a path to tranform it into a real integral in t. But now we have a real integral in t which we rewrite as an integral in z.

Ex: $F = \cos^3(t) + \sin^2(t)$.

$$F((z + 1/z)/2, (z + 1/z)/2i)\frac{1}{iz} = \frac{1}{8i}z^2 - \frac{1}{4i}z - 3i + \frac{1}{2iz} + \frac{3}{iz^2} - \frac{1}{4iz^3} + \frac{1}{iz^4}.$$

It is already a Laurent series expanded around $z = 0$, where it has a fourth order pole. There are no other poles. The residue is the coefficient of $1/z$, $1/(2i)$. So the integral is $2\pi i/(2i) = \pi$.

Trick: Real integrals over the whole real axis.

$$\int_{-\infty}^{\infty} f(x)dx \equiv \lim_{x_1\to-\infty, x_2\to\infty} \int_{x_1}^{x_2} f(x)dx.$$

This is not (necessarily) the same as the *Cauchy principal value* of the integral

$$\lim_{R\to\infty} \int_{-R}^{R} f(x)dx \equiv \mathrm{P}\int_{-\infty}^{\infty} f(x)dx.$$

If both exist, they are equal, but sometimes the second one exists, while the first one does not.

Trick: Closing the contour above.
If f differentiable in the upper half plane $\mathrm{Im}(z) \leq 0$ except for some poles, that are not on the real axis and f is such that along the half circle $s_R(t) = Re^{it}$, $t \in [0, \pi]$, $|f(z)| < A/R^2$, at large enough R for some constant A, then

$$\int_{-\infty}^{\infty} f(x)dx = 2\pi i\Sigma,$$

summing residues of poles in the upper half plane. The reason is that for a given R, we have that the path consisting of $[-R, R]$ and s_R is closed and runs counterclockwise. So we can sum the poles inside, and as $R \to \infty$, the s_R part goes to zero $\leq \pi RA/R^2$. Because also $f(x) < 1/R^2$ along the real axis, we know that the integral is convergent, and so calculating the principal value is sufficient.

Ex.

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)}, \qquad 0 < a \neq b > 0,$$

has single poles at $z = \pm ia$ and $z = \pm ib$. The residues at the upper half-plane poles are

$$\lim_{z\to ia} \frac{z-ia}{(x^2+a^2)(x^2+b^2)} = \frac{1}{2ia(b^2-a^2)}, \qquad \lim_{z\to ib} \frac{z-ib}{(x^2+a^2)(x^2+b^2)} = -\frac{1}{2ib(b^2-a^2)} \to 2\pi i\Sigma = \frac{\pi}{ab(a+b)}$$

This works, because $|f(z)| < \frac{1}{R^4}$, along s_R for R large enough.

Trick: Closing the contour below.
Similarly, we could have added the halfcircle $s_{-R} = -Re^{it}$ to $[-R, R]$. This however gives a closed path in the clockwise direction, so we have instead under the same conditions on f that

$$\int_{-\infty}^{\infty} f(x)dx = -2\pi i\Sigma,$$

where we now sum over the residues of poles in the lower half plane.

Ex.

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)}, \qquad 0 < a \neq b > 0,$$

has single poles at $z = \pm ia$ and $z = \pm ib$. The residues at the lower half-plane poles are

$$\lim_{z\to -ia} \frac{z+ia}{(x^2+a^2)(x^2+b^2)} = -\frac{1}{2ia(b^2-a^2)}, \qquad \lim_{z\to -ib} \frac{z+ib}{(x^2+a^2)(x^2+b^2)} = +\frac{1}{2ib(b^2-a^2)} \to -2\pi i\Sigma = \frac{\pi}{ab(a+b)}$$

Which of course gives the same result.

Ex. We could also have a complex function on the real axis

$$f(z) = \frac{e^{iz}}{(z^2+a^2)(z^2+b^2)}.$$

Then on s_R, $|f(z)| < e^{-y}/R^4$, where $y > 0$, since we are in the upper half plane. The poles are the same, and the residues are

$$\frac{e^{-a}}{2ia(b^2-a^2)}, \qquad -\frac{e^{-b}}{2ib(b^2-a^2)},$$

so we have

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{(x^2+a^2)(x^2+b^2)}dx = \int_{-\infty}^{\infty} \frac{\cos(x)+i\sin(x)}{(x^2+a^2)(x^2+b^2)}dx = \frac{\pi}{b^2-a^2}\left(\frac{e^{-a}}{a} - \frac{e^{-b}}{b}\right).$$

Comparing real and imaginary parts, we immediately have the result for two real integrals, one of which is zero. Note that here we could not have closed the contour below, since then $y < 0$, and the integral along s_{-R} would not go to zero. Obviously, if $e^{ix} \to e^{-ix}$, it would have been the other way around.

Trick: Rectangular contour.
What if we only have that $|f(z)| < A/R$? Then we can only calculate integrals of the form

$$\int_{-\infty}^{\infty} e^{ix} f(x)dx$$

using a different contour, which is

$$[-X_1; X_2], \quad [X_2; X_2+iY], \quad [X_2+iY, -X_1+iY], \quad [-X_1+iY, -X_1],$$

where $X_{1,2}, Y$ are real. This a counterclockwise rectangle of size Y times X_1+X_2. We now need to show that the integral along the three "non-real" sides are zero in the limit $X_{1,2}, Y$ going to infinity. Along the vertical edges,

$$|\int f| < \int_0^Y \frac{A}{X_2}e^{-t}dt < \frac{A}{X_2}, \qquad |\int f| < \int_0^Y \frac{A}{X_1}e^{-t}dt < \frac{A}{X_1},$$

which goes to zero by assumption. Also along the upper horizontal edge

$$|\int f| < |\int_{-X_1}^{X_2} \frac{A}{Y}e^{-Y}dt| = \frac{A(X_1+X_2)}{Y}e^{-Y},$$

which also goes to zero by taking the $y \to \infty$ limit first. But then we can take the $X_{1,2}$ limits separately, so the integral is convergent.

Ex.

$$\int_{-\infty}^{\infty} \frac{e^{ix}x^3}{(x^2+a^2)(x^2+b^2)}dx = \frac{i\pi}{b^2-a^2}(b^2e^{-b} - a^2e^{-a}), \quad 0 < a \neq b > 0.$$

Equating real and imaginary parts we again get two real integrals, one of which is zero. Obviously, had we had $e^{ix} \to e^{-ix}$, we could have drawn a clockwise box in the lower half plane and added those residues up instead, with a minus sign.

Trick: Poles on the real axis.
If there are poles on the real axis itself, we need to choose a contour that "avoids" them, but in some limit reduces to the real axis. So to either the half circle of the rectangle above, we add small half circles $s_\epsilon(t) = x_0 + \epsilon e^{-it}$, $t \in [0, \pi]$ (note the direction) with x_0 the pole, so that when the contour is "closed above" the poles on the real axis are outside the contour. Hence the contour integral is zero. Then we can calculate a principal value of the integral

$$P \int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty, \epsilon\to 0} \left(\int_{-R}^{x_0-\epsilon} f(x)dx + \int_{R}^{x_0+\epsilon} f(x)dx \right) = -\lim_{\epsilon\to 0} \int_{s_\epsilon} f(z)dz.$$

This may exist as a limit, even though the original integral does not. We have to argue for convergence on a case-by-case basis.

Ex.

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{x} dx.$$

f has a pole at $x_0 = 0$, but for the rest, we can again use the rectangle contour, and we know that the three off-axis edges give zero contribution to the integral. We therefore calculate

$$\lim_{\epsilon\to 0} \int_{s_\epsilon} \frac{e^{iz}}{z} dz = \lim_{\epsilon\to 0} \int_{s_\epsilon} (1/z + \Phi(z))$$

$$= \lim_{\epsilon\to 0} \int_{s_\epsilon} 1/z + \text{Something proportional to } \epsilon = -i\pi.$$

Since the total integral around the whole contour is zero and three of the edges give zero, the small circle and the real axis contributions must add up to zero, and so

$$\mathrm{P} \int_{-\infty}^{\infty} \frac{e^{ix}}{x} dx = i\pi = \mathrm{P} \int_{-\infty}^{\infty} \frac{\cos(x)}{x} dx + i\mathrm{P} \int_{-\infty}^{\infty} \frac{\sin(x)}{x} dx.$$

But the imaginary part is in fact analytic at $x = 0$, and so we can dispense with the principal value label, and we have (because the function is even)

$$\int_{0}^{\infty} \frac{\sin(x)}{x} = \frac{\pi}{2}.$$

Trick: Integrals of the type

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{\Phi(e^x)} dx.$$

We use the rectangular contour

$$[-X_1; X_2], \quad [X_2; X_2 + i2\pi], \quad [X_2 + i2\pi, -X_1 + i2\pi], \quad [-X_1 + i2\pi, -X_1],$$

On the upper horizontal edge we write

$$-\int_{-X_2}^{X_1} \frac{e^{a(-t+2\pi i)}}{\Phi(e^{-t})} dt = -e^{2\pi i a} \int_{X_1}^{X_2} \frac{e^{ax}}{\Phi(e^x)} dx.$$

If the function Φ is such that the integral vanishes on the vertical edges for $X_{1,2} \to \infty$, we have

$$(1 - e^{2\pi i a}) \int_{-\infty}^{\infty} \frac{e^{ax}}{\Phi(e^x)} dx = 2\pi i \Sigma,$$

with Σ the sum of residues of the poles in the rectangle.

Ex.

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{e^{2x}+1} dx, \quad (0 < a < 1).$$

There are infinitely many poles, but only two of them are in the rectangle $z = -e^{i\pi a/2}/2$ and $z = -e^{i3\pi a/2}/2$. Adding these up, we get

$$... = \frac{\pi}{2\sin(\pi a/2)}.$$