Mathematical Methods

Lecture 5.
13/10-2008

1 Analytic continuation

Wherever a function is analytic, it can be represented by a power series with a convergence radius equal to the distance to the nearest singularity. Power series around different points give different series, but on the overlap of their discs of convergence, the two sums agree (as they must! It's the same function).
Analytic continuation: If f_1 is analytic on a domain S_1 and f_2 is analytic on S_2, and $f_1 = f_2$ on the (non-empty) overlap between S_1 and S_2, f_2 is a *direct analytic continuation* of f_1 to S_2 (and vice versa).
Ex. $\frac{1}{1-z}$ is a direct analytic continuation of $\sum_n z^n$, $|z| < 1$ to $\mathbb{C}$, $z \neq 1$.
Ex. $\frac{1}{1-z}$ is a direct analytic continuation of $\frac{1}{1-x}$, $x \in \mathbb{R}$, $x \neq 1$ to $\mathbb{C}$, $z \neq 1$.

We can set up a whole chain of f_n defined on domains S_n, where S_n overlaps with S_{n-1} and S_{n+1} and where f_n is equal to $f_{n\pm1}$ on those overlaps, respectively. Then for instance f_8 is an *analytic continuation* of f_1, even though there is no overlap between S_1 and S_8. We can even have a chain of domains that come back to the original domain S_1.

If f can be analytically continued from S_1 to $S_n = S_1$ but $f_n \neq f_1$, f is said to be *multivalued* or *multiform*. Otherwise it is *uniform*.
Ex. $f(z) = \sqrt{z}$ is multiform, since for every $z = re^{i\theta}$, we can choose

$$\sqrt{z} = \sqrt{r}e^{i\theta/2} \quad \text{or} \quad \sqrt{r}e^{i\theta/2+i\pi}.$$

But we can analytically continue using the domains

$$H_1 = \mathbb{C},\ \ \mathrm{Re}(z) > 0, \quad H_2 = \mathbb{C},\ \ \mathrm{Im}(z) > 0, \quad H_3 = \mathbb{C},\ \ \mathrm{Re}(z) < 0, \quad H_4 = \mathbb{C},\ \ \mathrm{Im}(z) < 0.$$

As we go in the chain H_1, H_2, H_3, H_4 we can then define a continuous function which agrees on the overlaps, but as we get back to H_1 the angle has shifted by π! If we go around one more time, we can again define a continuous function and as we get back to H_1, we are back to the original angle.
Ex. $f(z) = z^{1/n} = r^{1/n}e^{i\theta/n}$, $n \in \mathbb{N}$, a similar thing happens, but we have to go around the origin n times before we get back to the original value for the argument.
Ex. $f(z) = \log(z)$. In order to have a continuous value of the function

$$\log(z) = \log(r) + i\theta,$$

we can again use the domains $H_{1,2,3,4}$, but every time we go around the origin, we will pick up 2π. hence we never get back to the original value. Before we then defined the principal value to be

$$\mathrm{Log}(z) = \log(r) + i[\theta \pmod{2\pi}],$$

so that all the revolutions around the origin are "projected" back to the cut plane $\mathbb{C}_\pi$, with the arbitrary choice of interval $\theta \in]-\pi, \pi]$.

Multiform functions on $\mathbb{C}$ can instead be thought of as defined on a relevant *Riemann surface*, which amounts to copying the complex plane an appropriate number of times "cut and glue" these copies together so that the function is continuous. Where we cut is called a *branch*, the function on each copy $\mathbb{C}$ *Riemann sheet* is a *branch*. The end of the branch cut (if there is one) is the *branch point*. This is best illustrated by examples.

Ex. $f(z) = \sqrt{z}$. We had to go through the domains $H_{1,2,3,4}$ twice, so let us copy the complex plane an introduce $H_{5,6,7,8} = H_{1,2,3,4}$, and say that H_5 follows H_4 and H_1 follows H_8. Then we can define a single valued continuous (and differentiable) function on the chain $H_{1,2,3,4,5,6,7,8,1}$ which returns to its original value. It can be thought of as two "sheets" of paper each representing $\mathbb{C}$, but cut along the negative real axis and glued together so that as you go around the origin and get to the negative axis, you "move up one sheet", go around again and then "move down one sheet".

Ex. $f(z) = z^{1/n}$, $n \in \mathbb{N}$. Now we have $H_{1,\ldots,4n,1}$ as our sequence on n copies of the complex plane, which together are glued together as our n-sheet Riemann surface. Every time we move around the origin, we go up one sheet, except the last one, where we go back down $n-1$ sheets.

Ex. $f(z) = \log(z)$. We now need infinitely many sheets to define our Riemann surface, and as we go around the origin, we go up one level. If we go the other way around we go down one level. Note that it is our choice to place the cut at $\theta = \pi$ (negative real axis), so that that's where we go up and down. It is equally valid to put the cut at any other θ. Our choice fits with our definition of $\text{Log}(z)$.

Ex. $f(z) = z^\alpha$, $\alpha = a + ib \in \mathbb{C}$. Using our definition we have the n'th *branch* of the function

$$(re^{i\theta})^\alpha = \exp\left(\alpha(\log(r) + i\theta + i2\pi n)\right), \quad \theta \in]-\pi, \pi].$$

We have

$$|f(z)| = \exp\left(a\log(r) - (\theta + 2\pi n)b\right), \qquad \arg(z) = b\log(r) + a\theta + 2\pi n a.$$

a) If $b \neq$, different n (different sheets) give different modulus, and we need infinitely many sheets for our Riemann surface. b) If $b = 0$ and a is irrational, the modulus is rational, but there is no n so that na is an integer, which would mean that we have returned to the original domain. Again, we need infinitely many sheets. c) If $b = 0$ and a is rational, however, there is an n so that na is integer, and we need that number of sheets.

Integration on the Riemann surface. If we have a multiform function defined on a Riemann surface, we can integrate it along a contour $s(t)$ from z_0 to z_1 using the antiderivative at the endpoints

$$\int_s f = F(z_1) - F(z_0)$$

but taking into account that which sheets z_1 and z_0 are on.

Ex. $z^{1/5}$, $s(t) = (1+t)e^{it}$, $t \in [0, 6\pi]$. The Riemann surface has 5 sheets, and we need to go 3 times around the origin (note that there is no pole at origin, it's a *branch point*). Take the antiderivative $F(z) = 5z^{6/5}/6$, and plug in

$$F(s(6\pi)) - F(s(0)) = \frac{5}{6}\left((1+t)^{6/5}e^{36\pi/5} - 1\right).$$

which is not the same as

$$F((1+6\pi)) - F(1) = \frac{5}{6}\left((1+6\pi)^{6/5} - 1\right).$$

Ex. Let's calculate

$$\int_0^\infty \frac{x^a}{(1+x^2)^2}dx, \qquad 1<a<2.$$

Extending to the complex plane, we have poles at $z=\pm i$, with residues

$$r_1 = \frac{i}{4}i^a(a-1) = \frac{i}{4}e^{ia\pi/2}(a-1), \qquad r_2 = -\frac{i}{4}(-i)^a = \frac{i}{4}e^{i3a\pi/2}(a-1).$$

The function is however multiform since $a>1$. Let us choose the contour from $\rho>0$ on the real axis to $R>\rho$. Then along counterclockwise full circle of radius R. Back along the real axis from R to ρ. Along a clockwise circle with radius ρ. It is clear that the integrals along the circles will go to zero in the limit $R\to\infty$, $\rho\to 0$.

The trick is now that as we go along the first circle, we go up one Riemann sheet, and so as we get back to the real axis, we have picked up a phase 2π. Hence we have

$$\int_\rho^R \frac{x^a}{(1+x^2)^2}dx + \int_R^\rho \frac{(xe^{i2\pi})^a}{(1+x^2)^2}dx = 2\pi i(r_1+r_2).$$

We therefore have

$$\int_\rho^R \frac{x^a}{(1+x^2)^2}dx - e^{2\pi i a}\int_\rho^R \frac{x^a}{(1+x^2)^2}dx = 2\pi i(r_1+r_2).$$

and taking the limits, we have

$$\int_0^\infty \frac{x^a}{(1+x^2)^2}dx = -\frac{1}{4}(1+(-1)^a)(a-1)\frac{\pi}{\cos(a\pi/2)}.$$

What does it mean? We can uniquely extend a function in an analytic way, although for multiform functions we need a non-trivial Riemann surface to make everything well-defined, continuous and differentiable everywhere. But then all the machinery of integration and poles works, if we keep in mind when integration contours go from one sheet to another.

2 Conformal transformations

A *conformal transformation* is a mapping from $\mathbb{C}$ to $\mathbb{C}$ which conserves the angle between paths.

Theorem: If the derivative $s'(t)$ of the path $s(t)$ exists and is non-zero, then the tangent of s at $z_0 = s(t_0)$ makes an angle $arg(s'(t_0))$ (mod 2π) with the real axis.

Consider a differentiable function $f(x+iy) = u(x,y)+iv(x,y)$ and two paths $s_{1,2}(t)$, we have

$$\begin{aligned}\arg((f(s_1))'(t_0)) - \arg((f(s_2))'(t_0)) &= \arg(f'(z_0)) + \arg(s_1'(t_0)) - \arg(f'(z_0)) - \arg(s_2'(t_0)) \\ &= \arg(s_1'(t_0)) - \arg(s_2'(t_0)).\end{aligned}$$

so the angle between the paths and the image of the paths is conserved, if f is differentiable.

Theorem: A differentiable function is conformal everywhere where $f'(z)\neq 0$.

Ex. $f(z)=z^3$, $u(x,y)=x^3-3xy^2$, $v(x,y)=3x^2y-y^3$, and the paths $s_1(t)=1+it$, $s_2(t)=t+i$ meet at right angles at $t=1$. We have

$$s_1: \quad u'(t) = -6t, \quad v'(t) = 3-3t^2 \qquad s_2: \quad u'(t) = 3t^2-3, \quad v'(t) = 6t.$$

Taking the inner product as vectors, we have

$$u_1'(t)u_2'(t) + v_1'(t)v_2'(t) = 36t(1-t^2)_{t=1} = 0, \qquad z = f(s_1(1)) = f(s_2(1)) = -2(1-i).$$

Ex. $f(z) = 1/z$, $u(x,y) = x/(x^2+y^2)$, $-y/(x^2+y^2)$. A circle $s(t) = ce^{it}$ is mapped into the circle e^{-it}/c, so with the inverse radius and running the other way around the origin.

A particular set of conformal transformations are the *Möbius* mappings

$$f(z) = \frac{az+b}{cz+d}, \quad a,b,c,d \in \mathbb{C}, \quad ad - bc/neq0.$$

Composition of Möbius mappings gives a Möbius mapping. They map circles and straight lines into circles and straight lines.

Theorem: Every Möbius mapping can be obtained by composition of a *translation*, an *inversion*, a *magnification*, a *a rotation* and a *translation*, where we define: *translation* $f(z) = z + k$, $k \in \mathbb{C}$, *rotation* $f(z) = e^{i\theta}z$, $\theta \in \mathbb{R}$, *magnification* $f(z) = hz$, $h > 0$, *inversion* $f(z) = 1/z$.

If $f = u(x,y) + iv(x,y)$, u and v are solutions to Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2},$$

and hence harmonic. Since the lines $u(x,y)$ =constant and $v(x,y)$ =constant are orthogonal and f is conformal, so are the lines in x, y for which $u(x,y) =$ constant and $v(x,y)$ =constant. In the field of potential theory. These are called *equipotential lines* and *stream lines*, respectively. in fluid dynamics, the fluid flows along the stream lines.

What does it mean? Conformal mappings conserve angles, and all differentiable mappings are conformal. In physics appplications, we sometimes find symmetry under conformal transformations, in the same way as we find symmetry under rflection, translations, rotations, Lorentz transformations etc. Møbius mappings are a particular subset of conformal transformations, and are all the possible combinations of Rotations, Translations, Inversions, Magnifications.