Mathematical Methods

Lecture 8.
3/11-2008

1 PDEs

Differential equations for functions of multiple variables $f(\mathbf{x}, t)$ involve partial derivatives and a called *partial differential equations*. In physics, we usually consider functions of time t and one or more spatial coordinates $\mathbf{x}$. Let us concentrate on linear PDEs, for which a general equation in one spatial dimension could be written (I sometimes use the notation $\partial_t = \frac{\partial}{\partial t}$)

$$\left(f(x,t)\partial_t^2 + g(x,t)\partial_x^2 + h(x,t)\partial_t\partial_x + j(x,t)\partial_t + k(x,t)\partial_x + l(x,t)\right) u(x,t) = r(x,t).$$

We will assume that the functions f, g, h, j, k, l, r are analytic where they are defined, which will be of the form

$$t \in [a,b], \qquad x \in [c,d].$$

The generalisation to more spatial dimensions is obvious.

Rather than a general study of this type of equations (which is exceedingly hard), we will concentrate on some physically relevant cases.

- With $r = l = k = j = h = 0$ and $f = 1$, $g = -c^2$, we recover the *(homogeneous) Wave equation in 1+1 dimensions*,

  $$\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2},$$

  A priory, c is just a constant, but we will see that it is the speed of propagation of the waves, $c^2 = T/\rho$ in terms of the denisty ρ and the string tension T.

- With $r = l = k = h = f = 0$ and $j = 1$, $g = -c^2$ we instead get the *(homogeneous) Heat equation in 1+1 dimensions*,

  $$\frac{\partial u(x,t)}{\partial t} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2},$$

  Now $c^2 = K/\sigma\rho$, in terms of the specific heat σ, the thermal conductivity K and the density ρ.

- With $r = l = k = h = j = g = 0$ and $f = 1$, we get the *Laplace equation*, here written in 2 dimensions. There is no longer a time dimension (or the dependence on time is undetermined)

  $$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = 0,$$

- The inhomogeneous Laplace equation is known as the *Poisson equation*,

  $$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = r(x,y).$$

The equations have multiple solutions in general, and a set of boundary conditions is required to unambiguously determine a solution.

Because the equations are linear, if $u_1(x, ...)$, $u_2(x, ...)$ are solutions to the homogeneous version of each equation, then so is

$$u(x, ...) = au_1(x, ...) + bu_2(x, ...), \qquad a, b \in \mathbb{C}.$$

1.1 Separation of variables and factorization

In many cases, functions of multiple variables are *separable* and the solution *factorizes* into product of single-variable functions. So in case the separation works in Cartesian coordinates, we have for instance

$$u(x, t) = F(x)J(t), \qquad u(x, y) = F(x)G(y), \qquad u(x, y, t) = F(x)G(y)J(t), \qquad u(x, y, z, t) = F(x)G(y)H(z)J(t).$$

when there is cylindrical or spherical symmetry, we may be able to separate in the corresponding variables

$$u(r, \theta, z, t) = F(r)G(\theta)H(z)J(t), \qquad u(r, \theta, \phi, t) = F(r)G(\theta)H(\phi)J(t).$$

For a linear homogeneous differential equation,

$$Du = 0,$$

we can hope to find some *eigenfunctions* u_n, with

$$Du_n = \lambda_n u_n.$$

If so, we can expand our solution and write

$$u = \sum_n c_n u_n, \quad \rightarrow \quad Du = \sum_n c_n \lambda_n u_n = 0,$$

which if the u_n form an orthonormal basis amounts to n separate equations to solve. Boundary conditions will give the coefficients c_n.

Ex.: Consider the operator $D = \partial/\partial x$. The eigenfunctions for this are the complex exponentials, since

$$\frac{\partial}{\partial_x} u = \frac{\partial}{\partial_x} e^{inx} = ine^{inx} \quad \rightarrow \quad \lambda_n = in.$$

Obviously, the solution to $Du = 0$ is $u = c_0 e^{i0x} = c_0$.

Ex.: Now consider the wave equation,

$$D = \partial_t^2 - c^2 \partial_x^2, \quad \rightarrow \quad u_n = e^{i\omega_n t + inx}, \quad \lambda_n = -\omega_n^2 + c^2 n^2.$$

To solve $Du = 0$, we require $\omega_n = \pm cn$, and so the complete solution is

$$u = \sum_n c_n e^{in(x-ct)} + \tilde{c}_n e^{in(x+ct)}.$$

More about this later. The boundary conditions tell us the allowed values for n and the values of c_n.

Ex.: The heat equation has

$$D = \partial_t - c^2\partial_x^2, \quad \rightarrow \quad u_n = e^{i\omega_n t + inx}, \quad \lambda_n = i\omega_n + c^2 n^2,$$

and we have

$$u = \sum_n c_n e^{inx - c^2 n^2 t)}.$$

Ex.: For general coefficient functions $f, g, h, \ldots$, it is hard to find a set of eigenfunctions e_n, so that

$$(e_m, De_n) \propto \delta_{mn}.$$

A general differential operator will mix up the basis both as a function of space and time. below we will see a few cases that work.

1.2 The Laplacian

In many applications, we encounter second order spatial derivatives in 1, 2 or 3 dimensions. Sometimes the problem at hand allows us to identify rotational or cylindrical symmetry, and it may be advantageous to change coordinate system for the spatial coordinates.

The *laplacian in cartesian coordinates* (x, y) in 2 dimensions

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

In *polar coordinates* ($r = \sqrt{x^2 + y^2}, \theta = \arctan(y/x)$) in 2 dimensions

$$\nabla^2 f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}$$

The *laplacian in cartesian coordinates* (x, y, z) in 3 dimensions

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2},$$

In *cylindrical coordinates* ($r\sqrt{x^2 + y^2}, \theta = \arctan(y/x), z = z$) in 3 dimensions

$$\nabla^2 f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2},$$

In *spherical coordinates* $x = r\cos\theta\sin\phi, y = r\sin\theta\sin\phi, z = r\cos\phi$,

$$\nabla^2 f = \frac{\partial^2 f}{\partial r^2} + \frac{2}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial f}{\partial \theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}.$$

Ex.: In quantum mechanics, the spatial derivative is the momentum operator,

$$i\hbar\frac{\partial}{\partial x} = p_x,$$

and hence the Laplacian encodes the kinetic energy,

$$-\frac{\hbar^2}{2m}\nabla^2 = -\frac{|\mathbf{p}|^2}{2m}.$$

1.3 Spherical harmonics

The angular part of the spherical Laplacian in 3 dimensions has a set of eigenfunctions known as *spherical harmonics*

$$\nabla^2 Y_{lm}(\theta,\phi) = -\frac{l(l+1)}{r^2} Y_{lm}(\theta,\phi).$$

These turn out to be an orthogonal basis for all square integrable functions defined on the unit sphere. Therefore if we assume that a solution factorizes,

$$u(r,\theta,\phi) = f(r)g(\theta,\phi),$$

we can always expand it as ($l = 0,..,\infty$, $m = -l,..,l$)

$$u(r,\theta,\phi) = \sum_{lm} f(r) g_{lm} Y_{lm}(\theta,\phi),$$

with

$$g_{lm} = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin(\theta) g(\theta,\phi) Y_{lm}(\theta,\phi),$$

and we have

$$\nabla^2 u(r,\theta,\phi) = \sum_{lm} \left(\frac{d^2 f(r)}{dr^2} + \frac{2}{r}\frac{df(r)}{dr} - \frac{l(l+1)}{r^2} f(r) \right) g_{lm} Y_{lm}(\theta,\phi).$$

The spherical harmonics are *associated Legendre polynomials*,

$$Y_{lm}(\theta,\phi) = \sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}} e^{im\phi} P_{lm}(\cos(\theta)), \quad P_{lm}(\cos(\theta)) = \left[(1-x^2)^{m/2} \frac{d^m}{dx^m}(P_l(x)) \right]_{x=\cos(\theta)}$$

in terms of the normal Legendre polynomials with the argument $\cos(\theta)$. Note that in case there is no ϕ-dependence (cylindrical symmetry), only the $m = 0$ contributions survive, and we have

$$Y_{l0}(\theta,\phi) = \sqrt{\frac{(2l+1)}{4\pi}} P_l(\cos(\theta)).$$

For polar coordinates in 2 dimensions, we also have the notion of spherical harmonics, since the periodic solutions of

$$\nabla^2 g(\theta) = \frac{-n^2}{r^2} g(\theta),$$

are simply $\cos(n\theta)$, $\sin(n\theta)$, $n > 0$. Or equivalently $e^{in\theta}$, which is of course the three dimensional spherical harmonics, with the θ dependence taken out. They are orthonormal with the usual innner product.

Using this, and writing $u(r,\theta) = f(r)g(\theta)$ the radial equation becomes

$$\nabla^2 f(r) g(\theta) = \left(\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} - \frac{n^2}{r^2} f \right) g(\theta).$$

1.4 Boundary conditions

For *Cartesian coordinates*, we will usually assume that we are on a finite interval $t \in [0; t_f]$, $x, y, z \in [0; l]$, with boundary conditions

- *Dirichlet boundary conditions*:

  $$u(x, 0) = u_0(x), \qquad u(x, t_f) = u_f(x), \qquad \text{or} \qquad u(0, t) = u_0(t), \qquad u(l, t) = U_l(t),$$

  which in the case of time evolution means that we know the beginning and end configuration of the "field" $u(x, t)$. Or in the case of two spatial coordinates, it means we know the function around the whole edge of the area of definition. In many cases we will use the *trivial Dirichelt boundaries*, where we put the $u = 0$ along (parts of) the boundary.

- *von Neumann boundary conditions*:

  $$\frac{\partial f(x,t)}{\partial t}|_{t=0} = h_i(x), \qquad \frac{\partial f(x,t)}{\partial t}|_{t=t_f} = h_f(x),$$

  in which case we know the initial and final derivatives (or velocities).

- *Cauchy boundary conditions*:

  $$f(x, 0) = g(x), \qquad \frac{\partial f(x,t)}{\partial t}|_{t=0} = h(x),$$

  which are the most common in initial values problems, such as the evolution of equations of motion in time. Only information at the initial time is assumed.

- *Mixed boundary conditions*: Combinations of these. In a sense, Cauchy initial conditions are a mixture of Dirichlet and Neumann.

In the case of *spherical (in 3D) coordinates*, we instead have $\phi \in [0; 2\pi]$, $\theta \in [0; \pi]$, $r \in [0, a]$ (where $a = \infty$ is allowed). Then a boundary condition (Dirichlet, Neumann or mixed) is often specified on some sphere $x^2 + y^2 + z^2 = R^2$. Similarly for polar coordinates in 2D.

1.5 Laplace's equation in 2 dimensions

In 2 spatial dimensions, Laplace's equation coincides with the equation for analytic or harmonic functions, with the interpretation of x, y as the real and imaginary part of the argument $z = x + iy$,

$$\partial_x^2 u(x, y) + \partial_y^2 u(x, y) = 0.$$

We therefore need to find a complex differentiable function, of which $u(x, y)$ is the real (or imaginary) part. Given a boundary condition, say the function along the real axis x (or indeed any curve), the solution is the (real part of the) analytic continuation to the 2-dimensional "complex" plane.

In particular, in the spherically symmetric case, and if the boundary is a circle of radius R with $u(R, \theta) = f(\theta)$, we have Poisson's integral equation

$$u(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} f(\theta') \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - \theta') + r^2} d\theta'.$$

This follows directly from the fact that an analytic function is uniquely determined within a closed curve by the function on that curve.

Also, as we saw above, the have the radial equation

$$\left(\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} - \frac{n^2}{r^2} f\right) = 0,$$

for which the solution is

$$f(r) = c_1 \cosh(n \ln(r)) + c_2 \sinh(n \ln(r)).$$

1.6 Laplace's equation in 3 dimensions

In three dimensions, there is no possible link to analytic functions, although solutions are still called harmonic. Because of the rotational symmetry of the Laplacian, we stand the best chance assuming that we can factorize in spherical coordinates, and we have as above

$$\nabla^2 u(r,\theta,\phi) = \sum_{lm} \left(\frac{d^2 f(r)}{dr^2} + \frac{2}{r}\frac{df(r)}{dr} - \frac{l(l+1)}{r^2} f(r)\right) g_{lm} Y_{lm}(\theta,\phi) = 0.$$

Assuming again that our boundary is a sphere of radius R, $u(R,\theta,\phi) = h(\theta,\phi)$, we can expand on spherical harmonics

$$h(\theta,\phi) = \sum_{lm} h_{lm} Y_{lm}(\theta,\phi),$$

for some h_{lm}. Then we need to solve

$$\left(\frac{d^2 f(r)}{dr^2} + \frac{2}{r}\frac{df(r)}{dr} - \frac{l(l+1)}{r^2} f(r)\right) = 0,$$

for each value of l. This is a Cauchy equation, for which the solutions are

$$f(r) = r^l, \qquad f(r) = r^{-l-1}.$$

If we for simplicity assume that there is cylindrical symmetry, the complete solution is

$$u(r,\theta,\phi) = \sum_l A_l r^l P_l(\cos(\theta)), \quad r < R,$$

and

$$u(r,\theta,\phi) = \sum_l B_l r^{-l-1} P_l(\cos(\theta), \quad r > R,$$

where

$$A_l = \frac{2l+1}{2R^l}\int_0^\pi g(\theta) P_l(\cos(\theta))\sin(\theta)d\theta, \qquad B_l = \frac{(2l+1)R^{l+1}}{2}\int_0^\pi g(\theta) P_l(\cos(\theta))\sin(\theta)d\theta,$$

respectively.

Ex.: Solving the Laplace equation is sometimes known as *potential theory*, since solutions are *conservative potentials*. This means that given a solution $u(x,y,z)$, we can interpret it as the (gravitational of electromagnetic) potential that a particle moves in. Then, the force is the gradient of the potential

$$\mathbf{F}(x,y,z) = -\nabla u(x,y,z).$$

Not all sets of forces derive from a potential, but those that do are conservative in the sense the work needed to move a particle from one position to another does not depend on the trajectory along which we do it. In 2D this is precisely the Cauchy theorem, namely that if a function has an anti-derivative, integrals along a path only depend on the endpoints of that path. This also holds for functions who have an anti-derivative, ie, solutions to Laplace's equation.

Ex.: The gravitational field around a point mass has the potential

$$u(r) = -\frac{k}{r} + c, \qquad k > 0.$$

It obeys Laplaces equation in 3D, since

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} = 0$$

(there is no angular dependence). In fact, the gravitational field in D spatial dimensions has to obey

$$\frac{\partial^2 u}{\partial r^2} + \frac{D-1}{r}\frac{\partial u}{\partial r} = 0,$$

which means that

$$u(r) = \frac{k}{r^{D-2}} + c, \quad D \neq 2, \quad u(r) = k\ln(r) + c, \quad D = 2$$

usually, one fixed the constant by imposing $\lim_{r\to\infty} = 0$ and some value $-k$ at $r = R$. Then

$$u(r) = \frac{-kR^{D-2}}{r^{D-2}}.$$

Back in D=3, we have the force

$$F_r(r) = -\frac{k}{r^2}, \quad k = GmM$$

so that gravitation in attractive, and where we have used our knowledge of Newtonian gravity to fix the integration constant k.

It is important to note that Laplaces equation only holds at point were there is no mass (or electric charge in the case of electromagnetism). Otherwise we would have to solve the Poisson equation, for which solutions do not superpose.