# Mathematical Methods

**Lecture 9.**
**10/11-2008**

## 1 Differential equations involving the Laplacians

Let us reiterate, that if we have a homogeneous differential equation involving the Laplacian,

$$Du = \left(D_t + \nabla^2\right) u = 0,$$

we are looking for a set of eigenfunctions of the differential operator to expand on.
Often we have the case when the function factorizes in Cartesian coordinates,

$$u(t,x,y,z) = f(t)g(x,y,z),$$

and we can use the eigenfunctions of the differential operator to expand on,

$$g(x,y,z) = \sum_k c_k u_{\mathbf{k}}, \quad u_{\mathbf{k}}(x,y,z) = e^{ik_x x + ik_y y + ik_z z},$$

with eigenvalues

$$\nabla^2 u_{\mathbf{k}}(x,y,z) = |\mathbf{k}^2| u_{\mathbf{k}}(x,y,z).$$

In order to solve the differential equation, we need to find some solutions to

$$D_t f_{\mathbf{k}}(t) = -|\mathbf{k}^2| f_{\mathbf{k}}(t),$$

so eigenfunctions of $D_t$. Then the combination

$$f_{\mathbf{k}}(t) u_{\mathbf{k}}(x,y,z)$$

will solve the equation. This applies in 1,2 and 3 spatial dimensions.
The other important case is when there is factorization in spherical (or polar) coordinates. Then we write

$$\nabla^2 u(t,r,\theta\phi) = \nabla^2 f(t)g(r)h(\theta,\phi) = f(t)\sum_{lm}(\partial_r^2 + \frac{2}{r}\partial_r - \frac{l(l+1)}{r^2})g_k^l(r)Y_{lm}(\theta,\phi).$$

which means we need to find the eigenfunctions satifying

$$(\partial_r^2 + \frac{2}{r}\partial_r - \frac{l(l+1)}{r^2})g_k^l(r) = k^2 g_k^l(r),$$

for each $l$ and $k$. Then to solve the equation, we have to find other eigenfunctions

$$D_t f_k(t) = -k^2 f_k(t),$$

in which case, the combinations

$$f_k(t)\sum_{lm} g_k^l(r) Y_{lm}(\theta,\phi)$$

are solutions to the equation in spherical coordinates.

## 2 The wave equation

### 2.1 Wave equation in 1+1 dimensions

We have

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2},$$

Assuming factorization $u(x,t) = F(t)G(x)$, we have

$$0 = \frac{\partial^2 F(t)}{\partial t^2} G(x) - c^2 F(t) \frac{\partial^2 G(x)}{\partial x^2} \quad \rightarrow \quad \frac{1}{c^2 F(t)} \frac{\partial^2 F(t)}{\partial t^2} = \frac{1}{G(x)} \frac{\partial^2 G(x)}{\partial x^2} = -k^2.$$

where in the second equation the fact that one side is independent of $t$ and the other of $x$ means that they must both be constant $-k^2$. We thereby reduce the problem to the two second order homogeneous ODEs for each value of $k$

$$\frac{\partial^2 F_k(t)}{\partial t^2} = -c^2 k^2 F_k(t), \qquad \frac{\partial^2 G_k(x)}{\partial x^2} = -k^2 G_k(x).$$

The solution to the $G$ equation is

$$G_k(x) = e^{\pm ikx}, \quad k > 0.$$

**Spatial boundaries** If we assume that we are on a half-interval $x \in [0; l]$, with trivial Dirichlet boundaries we fix the possible values of $k$ so that we have

$$G_k(x) = c_n e^{i\frac{n\pi x}{l}} + c_{-n} e^{-i\frac{n\pi x}{l}}, \qquad -k^2 = -(n\pi/l)^2,$$

with $n$ integer. We can choose to use only half of the degrees of freedom and write

$$G_n(x) = d_n \sin\left(\frac{n\pi x}{l}\right).$$

**Ex.:** We note that our choice of sign of the constant $(-k^2)$ was informed by our boundary conditions. The equation

$$\partial_x^2 G(x) = k^2 G(x),$$

has solutions

$$\tilde{G}_k(x) = e^{\pm kx},$$

but these are not periodic and cannot obey the Dirichlet boundary conditions, so we discard them. In physical applications, the relevant sign is often $-k^2$, and the trigonometric functions will give us the waves we expect. However, we should keep in mind that "an imaginary momentum" $ik$ can sometimes occur, and will give an exponentially decaying/growing solution.

**Eigenfunctions** Continuing with the trigonometric case, we need to solve the $F$ equation with the same eigenvalues $-k^2$, in general

$$F_k(t) = e^{\pm ikt},$$

and with our $x$-boundary conditions, these are restricted to

$$F_n(t) = A_n e^{i\frac{cn\pi t}{l}} + B_n e^{-i\frac{cn\pi t}{l}}.$$

Therefore a basis of solutions is

$$u_n(x,t) = G_n(x)F_n(t) = \left(A_n e^{i\frac{cn\pi t}{l}} + B_n e^{-i\frac{cn\pi t}{l}}\right)\sin\left(\frac{n\pi x}{l}\right),$$

known as *eigenfunctions* with eigenvalues $\lambda_n = cn\pi/l$. We note however that $Du_n = 0$, and *not* $Du_n = \lambda_n u_n$.

**Ex.:** These are the *modes* on a string, such as for a guitar or a violin. The $n = 1$ case is the base note, $n = 2$ is the octave, $n = 3$ the octave plus fifth. In general, multiplying $n$ by 2 gets the next octave, whereas multiplying by 3/2 gets the next fifth up. Since on a keyboard a fifth is 7 semi-tones and an octave is 12, one would expect that after 84 semi-tones, the two progressions should coincide. But in fact, $2^7 = 128$ (seven octaves) and $(3/2)^{12} = 129.75$ (12 fifths). So there is a slight mismatch. In practice this is smeared out by tuning slightly off pitch, known as a tempered tuning. Some old organs are tuned specifically in one key, but then one cannot play in some other keys.

**Boundaries in time:** Let us add Cauchy boundary conditions for time to the trivial Dirichlet boundary conditions in space

$$u(x,0) = g(x), \qquad \frac{\partial u(x,t)}{\partial t}|_{t=0} = h(x), \qquad G(0,t) = G(l,t) = 0.$$

This gives us

$$u(x,0) = g(x) = \sum_n u_n(x,0) = \sum_n (A_n + B_n)\sin\left(\frac{n\pi x}{l}\right).$$

Similarly,

$$\frac{\partial u(x,t)}{\partial t}|_{t=0} = h(x) = \sum_n (A_n - B_n) i\frac{cn\pi}{l}\sin\left(\frac{n\pi x}{l}\right).$$

These two conditions fix the coefficients $A_n$ and $B_n$ in terms of the Fourier coefficients of $g(x)$ and $h(x)$,

$$(A_n + B_n) = \frac{2}{l}\int_0^l dx\, g(x)\sin\left(\frac{n\pi x}{l}\right), \qquad i\frac{cn\pi}{l}(A_n - B_n) = \frac{2}{l}\int_0^l dx\, h(x)\sin\left(\frac{n\pi x}{l}\right),$$

We could have chosen von Neumann boundaries is space, in which case we would have

$$u(x,0) = g(x), \qquad \frac{\partial u(x,t)}{\partial t}|_{t=0} = h(x), \qquad \frac{\partial u(x,t)}{\partial x}|_{x=0} = \frac{\partial u(x,t)}{\partial x}|_{x=l} = 0,$$

But now it is not obvious how to find a good basis to expand on!

For general Dirichlet boundary conditions in space, we have

$$u(x,0) = g(x), \qquad \frac{\partial u(x,t)}{\partial t}|_{t=0} = h(x), \qquad G(0) = l_1(0,t), \qquad G(l) = l_2(0,t),$$

and the $x$-basis is no longer time-independent. It is unlikely that the factorization holds, although of course there is a solution, which one could for instance find numerically.

## 2.2 Coordinate change and D'Alembert's solution

As a trick, we can change coordinates to $v_\pm = x \pm ct$. Then the wave equation becomes simply

$$\frac{\partial^2 u(v_+, v_-)}{\partial v_+ \partial v_-} = 0,$$

which means that

$$u(v_+, v_-) = g_+(v_+) + g_-(v_-).$$

With the boundary conditions,

$$u(x, 0) = f(x), \qquad u'(x, 0) = g(x).$$

we conclude that the general solution to the wave equation is of the form

$$u(x,t) = \frac{1}{2}\left(f(x+ct) + f(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds,$$

Note that in the case of $g(x) = 0$, the solution is given by making the replacement $f(x) \to f(x \pm ct)$. This corresponds to a right- and left-moving wave with velocity $c$, so that at time $t_2$, the field value at $x_2$ is

$$u(x_2, t_2) = f(x_1 + ct_1) + f(x_1 - ct_1), \qquad x_1 \pm ct_1 = x_2 \pm ct_2.$$

If $t_2 > t_1$ then the first term is to the left of $x_2$ and the second to the right. So the first term is a right-moving and the second a left-moving wave.

**Ex.:** The solutions we found before are of this type, since

$$\left(c_k e^{ickt} + c_{-k} e^{-ickt}\right)\left(e^{ikx} + e^{-ikx}\right) = \left(c_k e^{ik(x+ct)} + c_{-k} e^{-ik(x+ct)}\right) + \left(c_{-k} e^{ik(x-ct)} + c_k e^{-ik(x-ct)}\right).$$

**Ex.:** The trick of changing coordinates can be used more generally, so simplify a partial differential equation, although sometimes it requires an inspired choice of coordinates. If the equation is of the form

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} = 0$$

we can think of it as the matrix

$$\begin{pmatrix} A & B \\ B & C \end{pmatrix}$$

which we can simplify by a change of coordinates. In the case above, we made the coordinate change (setting $c = 1$ for the moment)

$$x, t \to v_+ = x + t, v_- = x - t,$$

which amounts to the transformation

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$$

leaving only the off-diagonal equation

$$4\frac{\partial^2 u(v_+, v_-)}{\partial v_+ \partial v_-} = 0.$$

Note also that

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2} = \left(\frac{\partial}{\partial x} - c\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial x} + c\frac{\partial}{\partial t}\right) u(x,t) = c^2 \frac{\partial}{\partial v_-}\frac{\partial v_+}{\partial} u(x,t).$$

## 2.3 Wave equation in 2+1 dimensions

The wave equation of course generalizes to more spatial dimensions,

$$\left(\frac{\partial^2}{\partial t^2} - c^2 \sum_i \frac{\partial^2}{\partial_{x_i}}\right) u(t, \mathbf{x}) = 0,$$

and we have the general solutions

$$G_{\mathbf{k}}(x,t) = e^{\pm i\mathbf{k}\mathbf{x} \pm i|\mathbf{k}|ct},$$

which is the generalisation of D'Alembert's solution to more dimension. It simply encodes, that there is a direction $\mathbf{k}/|\mathbf{k}|$, along which a forward and a backward wave propagate at the speed $c$.

Let us consider a finite rectangle in 2-dimensional space $x, y \in [0, l]$, and we will restrict ourselves to trivial Dirichlet boundary conditions in space, $u(0,y,t) = u(x,0,t) = 0$. We assume initial conditions in time in terms of

$$u(x,y,0) = f(x,y) \quad , u'(x,y,0) = g(x,y).$$

If we believe that the solution factorizes, we can write

$$u(x,y,t) = F(x)G(y)H(t),$$

and we immediately write down the eigenfunctions,

$$u_n(x,y,t) = \left(A_n e^{i\frac{\omega_n t}{l}} + B_n e^{-i\frac{\omega_n t}{l}}\right) \sin\left(\frac{\pi}{l} n_1 x\right) \sin\left(\frac{\pi}{l} n_2 y\right),$$

or

$$u_n(x,y,t) = \left(A_n e^{i\frac{\omega_n t}{l}} + B_n e^{-i\frac{\omega_n t}{l}}\right) \left(e^{i\frac{\pi}{l}(n_1 x + n_2 y)} + e^{-i\frac{\pi}{l}(n_1 x + n_2 y)}\right)$$

with $\omega_n^2 = c^2\pi^2(n_1^2 + n_2^2)/l^2$, and where

$$f(x,y) = \sum_n (A_n + B_n) \sin\left(\frac{\pi}{l} n_1 x\right) \sin\left(\frac{\pi}{l} n_2 y\right),$$

$$g(x,y) = \sum_n i\omega_n (A_n - B_n) \sin\left(\frac{\pi}{l} n_1 x\right) \sin\left(\frac{\pi}{l} n_2 y\right),$$

The coefficients $A_n$ and $B_n$ can again be found by Fourier decomposition.

## 2.4 Wave equation in polar coordinates

In 2 dimensions, the symmetries of the problem may not allow for factorization in Cartesian coordinates. Sometimes it helps to go to polar coordinates.

For simplicity, we can assume rotational symmetry, in which case there is no dependence on $\theta$ and using the polar form of the Laplacian, the wave equation reduces to

$$\frac{\partial^2 f(r,t)}{\partial t^2} = c^2 \left(\frac{\partial^2 f(r,t)}{\partial r^2} + \frac{1}{r}\frac{\partial f(r,t)}{\partial r}\right).$$

The equation is defined on a disc $r < R$, and we will assume the boundary conditions

$$f(R,t) = 0, \qquad f(r,0) = g(r), \qquad \frac{\partial f(r,t)}{\partial t}|_{t=0} = h(r).$$

We separate variables $f(r,t) = G(r)F(t)$, to find

$$\frac{\partial^2 G_k(r)}{\partial r^2} + \frac{1}{r}\frac{\partial G_k(r)}{\partial r} + k^2 G_k(r) = 0, \quad \frac{\partial^2 F_k(t)}{\partial t^2} + c^2 k^2 F_k(t) = 0.$$

By using $s = kr$, $ds = kdr$, we have Bessel's equation

$$\frac{\partial^2 G(s)}{\partial s^2} + \frac{1}{s}\frac{\partial G(s)}{\partial s} + G(s) = 0$$

with $\nu = 0$, which therefore has the general solution

$$G(kr) = c_1 J_0(kr) + c_2 Y_0(kr).$$

$Y_0$ is divergent at $r = 0$, which can in some cases be considered "unphysical" (but sometimes not!) and we have

$$G(kr) = J_0(kr).$$

We can enforce $f(R,t) = 0$ at any of the zeros of $J_0(kr)$, $\alpha_n$. This fixes a set of $k$

$$k_n = \alpha_n/R, \rightarrow G_n(r) = J_0\left(\alpha_n r/R\right).$$

Note that this is similar to enforcing different trigonometric modes, but now for a polar wave. The solution to the $F$ is the usual one

$$F_n(t) = A_n e^{ick_n t} + B_n e^{-ick_n t},$$

so that

$$f(r,t) = \sum_{n=0}^{\infty} \left(A_n e^{ick_n t} + B_n e^{-ick_n t}\right) J_0\left(\alpha_n r/R\right),$$

with the Bessel-Fourier coefficients

$$A_n + B_n = \frac{2}{R^2 J_1^2(\alpha_n)} \int_0^R r g(r) J_0\left(\alpha_n r/R\right) dr,$$

and

$$-ick_n(A_n - B_n) = \frac{2}{R^2 J_1^2(\alpha_n)} \int_0^R r h(r) J_0\left(\alpha_n r/R\right) dr.$$

## 2.5 Wave equation in 3+1 dimensions

Probably the most physically relevant case is to have 3 spatial dimensions, in which case the basis of solutions is again

$$G_{\mathbf{k}}(x,t) = e^{\pm i\mathbf{k}\mathbf{x} \pm i|\mathbf{k}|ct},$$

Imposing trivial Dirichlet boundaries in space as above, we have $\mathbf{k} = \mathbf{n}\pi/l$ and in the real case

$$f_{n1,n2,n3}(\mathbf{x},t) = G_{n_1}(x_1)G_{n_2}(x_2)G_{n_3}(x_3)F_n(t), \qquad G_{n_i}(x) = \sin\left(\frac{n_i \pi x_i}{l_i}\right),$$

and

$$F_n(t) = A_n e^{i\omega_n t} + B_n e^{-i\omega_n t},$$

with

$$\omega_n^2 = c^2\pi^2(n_1^2 + n_2^2 + n_3^2)/l^2.$$

We can again find the $A_n$ and $B_n$ by Fourier expansion.

### 2.6 Wave equation in spherical coordinates

In some cases, factorization is achieved by going to spherical coordinates $r, \theta, \phi$, and we have

$$u(t, r, \theta, \phi) = F(t)G(r)H(\theta, \phi),$$

say, where we assumed factorization in time and radial coordinate. In general, we would then proceed to expand on spherical harmonics as for the case of the 3 dimensional Laplace equation, but for simplicity, let us assume that there is rotational symmetry, and so we have

$$\left(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial r^2} - c^2\frac{2}{r}\frac{\partial}{\partial r}\right)F(t)G(r) = 0 \quad \rightarrow \quad \frac{\partial^2 F(t)}{c^2 F(t)\partial t^2} = \left(\frac{\partial^2 G(r)}{G(r)\partial r^2} + \frac{2}{rG(r)}\frac{\partial G(r)}{\partial r}\right) = -k^2.$$

The time-equation is the usual one with the solutions

$$F(t) = c_k e^{ikt} + c_{-k}e^{-ikt}.$$

The radial equation is no longer a Bessel equation, but has the solutions

$$G_k(r) = \frac{c_k e^{ikr} + \tilde{c}_k e^{-ikr}}{r}.$$

.

**Ex.:** As an example, we could have the boundary condition

$$\begin{aligned} u(R,t) = 0, \qquad &\rightarrow \quad G_k(r) = \tfrac{c_k e^{ikR} + \tilde{c}_k e^{-ikR}}{R} \\ &\rightarrow \quad c_k = -\tilde{c}_k, \quad kR = n\pi. \end{aligned}$$

Note that the solution is regular at $r = 0$, $\lim_{r\to 0} G_k(r) = c_k k$.

# 3 The heat equation in 1+1 dimension

The heat or Schrödinger equation is first order in time. As a result, when separating in variables, the spatial part is still acted on by the Laplacian and so has the same basis solutions as in the case of the wave equation. The time -solutions are however different.

After separation of variables, we have the two differential equations

$$\begin{aligned} \frac{\partial u(x,t)}{\partial t} &= c^2\frac{\partial^2 u(x,t)}{\partial x^2} \rightarrow \\ \frac{1}{c^2 F(t)}\frac{\partial F(t)}{\partial t^2} &= \frac{1}{G(x)}\frac{\partial^2 G(x)}{\partial x^2} = -k^2. \end{aligned}$$

In Cartesian coordinates, the spatial solutions are as above

$$G_k(x) = e^{\pm ikx}, \quad \text{or} \quad \sin(kx), \quad \cos(kx),$$

while the time solutions are instead

$$F_k(t) = e^{-k^2 t},$$

so that the eigenfunctions are

$$u_k(x,t) = e^{-c^2k^2t}\left(c_k e^{ikx} + c_{-k}e^{-ikx}\right)$$

We again impose trivial Dirichlet boundary conditions in space and "Cauchy" boundary conditions in time

$$f(0,t) = f(l,t) = 0, \qquad f(x,0) = g(x)$$

Note that because the equation is first order in time derivatives, we don't need a boundary condition on $\frac{\partial f(x,t)}{\partial t}$. The solution to the $G$ equation is

$$G_n(x) = \sin(\frac{n\pi x}{l}),$$

as above. The $F(t)$ equation now has solutions

$$F_n(t) = B_n \exp(-\lambda_n^2 t), \qquad \lambda_n = cn\pi/l.$$

Hence the complete solution is

$$f(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{l}\right) \exp(-\lambda_n^2 t),$$

where the $B_n$ are the Fourier coefficients of the initial condition

$$B_n = \frac{2}{l} \int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx.$$

In 2+1 or 3+1 dimensions the result generalises in the obvious way: Take the solution to the wave equation in Cartesian of spherical (polar) coordinates, and replace the time-dependent oscillating factor by

$$F_n(t) = B_n \exp(-\lambda_n^2 t), \qquad \lambda_n = c\sqrt{n_1^2 + n_2^2 + ...}\pi/l.$$