Mathematical Methods.

Problem set 1.
Hand-in 22/9-2008

1. Reduce:

$$(4-3i)+(2i-8) \qquad 3(-1+4i)-2(7-i)$$
$$(3+2i)(2-i) \qquad (i-2)(2(1+i)-3(i-1))$$
$$\frac{2-3i}{4-i} \qquad (4+i)(3+2i)(1-i)$$
$$\frac{(2+i)(3-2i)(1+2i)}{(1-i)^2} \qquad (2i-1)^2\left(\frac{2-i}{1+i}+\frac{4}{1-i}\right)$$
$$\frac{i^4+i^9+i^{16}}{2-i^5+i^{10}-i^{15}} \qquad 3\left(\frac{1+i}{1-i}\right)^2-2\left(\frac{1-i}{1+i}\right)^3$$

2. With $z_1 = 1-i$, $z_2 = -2+4i$, $z_3 = \sqrt{3}-2i$, calculate

$$z_1^2+2z_1-3 \qquad |2z_2-3z_1|^2$$
$$(z_3-z_3^*)^5 \qquad |z_1z_2^*+z_2z_1^*|$$
$$\frac{1}{2}\left(\frac{z_3}{z_3^*}+\frac{z_3^*}{z_3}\right) \qquad |\frac{z_1+z_2+1}{z_1-z_2+i}|$$
$$((z_2+z_3)(z_1-z_3))^* \qquad |z_1^2+z_2^{*,2}|^2+|z_3^{*,2}-z_2^2|^2$$
$$\mathrm{Re}(2z_1^3+3z_2^2-5z_3^2) \qquad \mathrm{Im}(z_1z_2/z_3)$$

3. Show that

$$(z_1z_2)^* = z_1^*z_2^* \qquad \left(\frac{z_1}{z_2}\right)^* = \frac{z_1^*}{z_2^*}$$

Generalize this to products and fractions of any number of z's.

4. Find all the roots of the following polynomia and sketch them in the complex plane

$$5z^2+2z+10 \qquad z^2+(i-2)z+(3-i)$$
$$z^4+z^2+1 \qquad z^5-2z^4-z^3+6z-4$$

5. Find the solutions of

$$z^n-1=0$$

for $n = 2,3,4,5,6,7,8$. Sketch them in the complex plane. These are called the *n'th roots of unity*.

6. Determine which of these sequences is convergent, and find their limits.

$$z_n = (1+i)^n \quad z_n = \frac{(1+i)^n}{n}, \quad z_n = \frac{(1+i)^n}{n!}, \quad z_n = \frac{1}{(1+i)^n}, \quad z_n = \frac{n}{(1+i)^n}, \quad z_n = \frac{n!}{(1+i)^n}.$$

7. Find the radius of convergence for the following series (for some $k \in \mathbb{N}$, $a \in \mathbb{C}$)

$$\sum_n z^n, \qquad \sum_n \frac{z^n}{n}, \qquad \sum_n \frac{z^n}{n!}, \qquad \sum_n n! z^n, \qquad \sum_n \frac{z^n}{n^k}, \qquad \sum_n a(a-1)..(a-n+1)\frac{z^n}{n!},$$

$$J_0(z) = \sum_n (-1)^n \frac{1}{(n!)^2}\frac{z^n}{2^n}, \qquad \text{(Bessel function of degree 0)},$$

$$z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + ..., \qquad 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + ..., \qquad z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + ...$$

8. With $f(z) = u(x,y) + iv(x,y)$, $z = x + iy$, explicitly write out u and v for

$$f(z) = 1/z, \quad D = \{z \in \mathbb{C} | z \neq 0\} \qquad f(z) = |z|, \quad D = \mathbb{C} \qquad f(z) = z^*, \quad D = \mathbb{C}.$$

Where do the Cauchy-Riemann equations hold?

9. Check the Cauchy-Riemann equations for the following u, v-functions, and find the differentiable functions they originate from:

$$\begin{aligned} u(x,y) &= x^3 - 3xy^2, & v(x,y) &= 3x^2y - y^3, \\ u(x,y) &= \sin(x)\cosh(y), & v(x,y) &= \cos(x)\sinh(y), \\ u(x,y) &= \frac{x}{x^2+y^2}, & v(x,y) &= \frac{-y}{x^2+y^2}, \quad (x^2+y^2) \neq 0, \\ u(x,y) &= \frac{1}{2}\log(x^2+y^2), & v(x,y) &= \sin^{-1}\left(\frac{y}{(x^2+y^2)^{1/2}}\right), \quad x > 0. \end{aligned}$$

10. Consider

$$f(z) = \frac{xy^2(x+iy)}{x^2+y^2}, \quad z \neq 0, f(0) = 0.$$

Show that

$$\lim_{z \to 0} \frac{f(z) - f(0)}{z - 0} = 0$$

along any straight line to the origin $z(t) = (a + ib)t$, $a, b, t \in \mathbb{R}$. Now take the limit along the curve $z(t) = t^3 + it$. How about $at^\alpha + ibt^\beta$? Is $f(z)$ complex differentiable?

11. For the functions f and paths s below, compute $f'(s(t))$, $s'(t)$ and $(f(s))'(t)$, and demonstrate $(f(s))'(t) = f'(s(t))s'(t)$.

$$\begin{aligned} f(z) &= z^2, & s(t) &= t^3 + it^4, \quad z \in \mathbb{C}, \quad t \in [0,1], \\ f(z) &= 1/z, & s(t) &= \cos(t) + i\sin(t), \quad z \neq 0, \quad t \in [0, 2\pi], \\ f(z) &= 1 + z + z^2 + z^3 + ..., & s(t) &= t + it^2, \quad |z| < 1, \quad t \in [0, 1/2], \end{aligned}$$