Mathematical Methods.

Problem set 10.
Hand-in 1/12-2008

1. Solve the one-dimensional heat equation on an interval $x \in [0, 10]$, with boundary conditions $u(x = 0, t) = u(x = 10, t) = 0$, with initial "temperature" profile

   $a)\ u(x, t = 0) = \sin(\pi x/10), \quad b)\ u(x, t = 0) = x,\ x < 5;\ u(x, t = 0) = 10 - x,\ x > 5, \quad c)\ u(x, t = 0) = x(100 - x^2).$

2. Solve the one-dimensional wave equation on an interval $x \in [0, 1]$, with boundary conditions $u(x = 0, t) = u(x = 1, t) = 0$, with initial velocity zero, and with initial profile

   $a)\ \ u(x, t = 0) = kx(1 - x), \qquad b)\ \ u(x, t = 0) = k \sin(2\pi x), \qquad c)\ \ u(x, t = 0) = k(x - x^3).$

3. Solve the one-dimensional wave equation on an interval $x \in [0, 1]$, with boundary conditions $u(x = 0, t) = u(x = 1, t) = 0$, with initial profiles as above, but with initial velocities (use a) with a) above, b) with b), c) with c))

   $a)\ \ \partial_t u(x, t = 0) = k \sin^2(\pi x), \qquad b)\ \ \partial_t u(x, t = 0) = k(x^2 - x^4), \qquad c)\ \ \partial_t u(x, t = 0) = kx(2 - 2x).$

4. Consider functions defined on $x \in [0, 1]$. Look up the axioms that define a vector space, and determine which of the following sets of functions constitute a vector space

   - a) All functions with two continuous derivatives,
   - b) All polynomials with $f(x = 0) = 0$.
   - c) All polynomials with $f(x = 0) = 1$.
   - d) All polynomials of degree 9.
   - e) All solutions to the differential equation (a,b,c real numbers)

     $$au_{xx} + bu_x + cu = 0.$$

   - f) All solutions to the differential equation

     $$u_{xx} + x^2 u = 0.$$

5. Consider the vector space spanned by the polynomials $1, x, x^2, x^3$. Consider also the operator D, which denotes normal differentiation, and the operator X, which multiplies by x. In the basis given, write a matrix representation of the combined operators

   $$a)\ \ D, \qquad b)\ \ XD - DX, \qquad c)\ \ X^2 D^2 - D^2 X^2.$$

6. Consider the integral operator with kernel $K(x, y) = \sin(x) \sin(y)$, $x, y \in [-\pi, \pi]$. Show that there are infinitely many linearly independent eigenvectors with eigenvalue zero.