Mathematical Methods.

Problem set 2.
Hand-in 29/9-2008

Well done on the first problem sheet! I admit it was a bit long.
This one looks long, too, but there is a lot of drawing involved, so it shouldn't be so bad.
It helps to remember the **Theorems** to find short-cuts to the answers.
The next problem sheet will be shorter...I promise!

1. **Fun with power series:** Of two power series, $s(z) = \sum a_n z^n$ and $c(z) = \sum b_n z^n$, we know only that

$$s'(z) = c(z), \qquad c'(z) = -s(z), \qquad s(0) = 0, \qquad c(0) = 1.$$

Determine all the a_n, b_n and prove that $(s(z))^2 + (c(z))^2 = 1$.

2. **More fun with power series:** The Bessel function of order n is

$$J_n(z) = \sum_r \frac{(-1)^r (z/2)^{n+2r}}{r!(n+r)!}.$$

Find the radius of convergence and show the following relations,

$$J_{n-1}(z) + J_{n+1}(z) = \frac{2n}{z} J_n(z), \qquad J_2(z) - J_0(z) = 2J_0''(z), \qquad (z^n J_n(z))' = z^n J_{n-1}(z),$$
$$J_n'(z) = \frac{n}{z} J_n(z) - J_{n+1}(z) = \frac{1}{2}\left(J_{n-1}(z) - J_{n+1}(z)\right) = J_{n-1}(z) - \frac{n}{z} J_n(z).$$

3. **Some useful functions:** Write out the following in the form $u + iv$, u, v real functions,

$$\exp(i), \qquad e^{2+i\pi}, \qquad \frac{1}{\exp(2+i\pi)},$$
$$\sin(i), \qquad \cos(i), \qquad \sinh(i),$$
$$\cosh(i), \qquad \cos(\pi/4 - i), \qquad \tan(1+i).$$

4. **Simple differentiation:** Differentiate the following functions,

$$\exp(z^2 + 2z), \qquad \frac{1}{\exp(z)}, \qquad \frac{\exp(z^2)}{\exp(z+1)},$$
$$\tan(z^2), \qquad \frac{\sinh(z+2)}{\exp(z^3)}, \qquad \sin(z)\cosh(z)\exp(z).$$

5. **Useful paths:** Draw the paths

$$\begin{aligned}
s(t) &= e^{-it}, & t &\in [0, \pi], \\
s(t) &= 1 + i + 2e^{it}, & t &\in [0, 2\pi], \\
s(t) &= z_0 + re^{-it}, & t &\in [0, 2\pi], \quad r > 0, \quad z_0 \in \mathbb{C} \\
s(t) &= t + i\cosh(t), & t &\in [-1, 1], \\
s(t) &= \cosh(t) + i\sinh(t), & t &\in [-1, 1].
\end{aligned}$$

6. **Now integrate along contours:** Draw the contours and calculate the integral

$$\int_{s_{1,2}} \mathrm{Re}(z)dz, \quad \text{along the contours} \quad s_1 = [0, i], \quad s_2 = [0, 1] + [1, i],$$

$$\int_{s_{1,2}} |z|dz, \qquad \text{along the contours} \quad s_1 = [-i, i], \quad s_2 = e^{it}, \;\; t \in [-\pi/2, \pi/2],$$

$$\int_{s_{1,2}} z^4 dz, \qquad \text{along the contours} \quad s_1 = (1+i)t, \;\; t \in [0, 1] \quad s_2 = [0, 1] + [1, 1+i].$$

7. **But if they are differentiable, then I just have to...:** Calculate $\int_s f$ along the contour $s(t) = e^{it}$, $t \in [0, \pi]$,

$$\frac{1}{z^2}, \qquad \frac{1}{z}, \qquad \cos(z),$$
$$\sinh(z), \qquad \tan(z), \qquad \exp(z)^3.$$

8. **Partial integration:** Assuming f, g have continuous derivatives in D and s is a contour from z_1 to z_2 in D, *partial integration* works

$$\int_s fg' = f(z_2)g(z_2) - f(z_1)g(z_1) - \int_s f'g.$$

With $s(t) = e^{it}$, $t \in [0, \pi/2]$, compute

$$\int_s z\sin(z)dz, \qquad \int_s z\cos(z), \qquad \int_s ze^{iz}dz, \qquad \int_s z^2\sin(z)dz.$$

9. **Figuring out the argument:** Compute the principal logarithms (Log) of,

$$1+i, \qquad \sqrt{3}/2 + i, \qquad (1+i)^3, \qquad (\sqrt{3}/2+i)^{243}, \qquad (1+i)^2(\sqrt{3}/2+i)^3,$$

10. **Almost as for real logarithms:** Show that

$$\mathrm{Log}(z_1 z_2) = \mathrm{Log}(z_1) + \mathrm{Log}(z_2) + 2n\pi, \quad z_{1,2} \neq 0,$$

What are the possible values of the integer n?

11. **No real need to calculate, just look:** Draw the paths, specify a continuous choice of argument along them and compute the winding number around z_0.

$$\begin{aligned}
s(t) &= 2e^{-it}, & t &\in [0; 4\pi], & z_0 &= 0,\\
s(t) &= 2e^{-it}, & t &\in [0; 4\pi], & z_0 &= 1,\\
s(t) &= 2e^{-it}, & t &\in [0; 4\pi], & z_0 &= 3i,\\
s(t) &= t + i(1-t), & t &\in [0; 1], & z_0 &= 0,\\
s(t) &= t + i(1-t), & t &\in [0; 1], & z_0 &= 1+i,\\
s(t) &= t + i(1-t), & t &\in [0; 1], & z_0 &= -i,\\
s(t) &= t + i(1-t), & t &\in [0; 1], & z_0 &= 10i,\\
s(t) &= t - 1 + it^2, & t &\in [-1; 1], & z_0 &= 0,
\end{aligned}$$