Mathematical Methods.

Problem set 3.
Hand-in 6/10-2008

The third problem set is somewhat shorter, and aims at practicing all the tricks you need to deal with residue calculations. Note that it is rarely useful to calculate Laurent series coefficients using the integral definitions. Rather one should try using geometric series, differentiation and integration of known series to make "educated guesses" about series, poles, etc.

1. **Comples powers and the Log:** For $z, \beta \in \mathbb{C}$, $z \neq 0$, define the principal value for z^β,
$$z^\beta = \exp(\beta \mathrm{Log}(z))$$
Compute (reduce to the form $re^{i\theta}$)
$$1^{\sqrt{2}}, \quad (-2)^{\sqrt{2}}, \quad i^i, \quad 2^i, \quad 1^{-i}, \quad (3-4i)^{1+i}, \quad (3+4i)^5.$$
For $a, z \in \mathbb{C}_\pi$, find
$$(z^\beta)' \qquad (a^z)'$$

2. **Finding Taylor series:** Taylor expand the following functions around $z_0 = 0$
$$\sin^2(z), \qquad z^2(z-2)^{-2}, \qquad (az+b)^{-1}, \quad b \neq 0,$$
$$\int_0^z e^{w^2} dw, \qquad \sin(z)/z, \ (z \neq 0), \quad 1, \ (z=0) \qquad \int_0^z \sin(w)/w \, dw.$$

3. **Finding Laurent series:** Find a Laurent expansions of the following around $z_0 = 0$
$$(z-3)^{-1}, \quad (z-a)^{-k}, \quad k = 1, 2, 3, ..., \quad \frac{1}{z(1-z)}, \quad \frac{1}{(z-a)(z-b)},$$
$$z^3 e^{1/z}, \quad \exp(z + 1/z), \quad \cos(1/z), \quad \exp(z^{-5}).$$
In which annuli are the expansions valid?

4. **Finding poles:** Find the poles of the functions
$$\frac{1}{z^2+1}, \quad \frac{1}{z^4+16}, \quad \frac{1}{z^4+2z^2+1}, \quad \frac{1}{z^2+z-1}.$$

5. **The geometric series is your friend...** Find Laurent expansions of
$$\frac{1}{(z-1)^2(z-2)} \quad \text{on} \quad 0 < |z| < 1,$$
$$\frac{1}{(z-1)^2(z-2)} \quad \text{on} \quad 1 < |z| < 2,$$
$$\frac{1}{(z-1)^2(z-2)} \quad \text{on} \quad 2 < |z| < 3,$$

6. **Finding singularities:** Find all singularities and the behaviour at infinity for
$$(z-z^3)^{-1}, \quad \frac{z^5}{(2-z)^2}, \quad (e^z-1)^{-1} - 1/z, \quad \cot(1/z).$$