Mathematical Methods.

Problem set 4.
Hand-in 13/10-2008

Now with sub-itemization, for a superior computing experience!

1. Find the residue $\mathrm{res}(f, z_0)$ when

$$\begin{aligned}
a)\ f(z) &= z^{-3}\sin(z), \quad (z \neq 0), \quad z_0 = 0,\\
b)\ f(z) &= e^z z^{-n-1}, \quad (z \neq 0), \quad z_0 = 0,\\
c)\ f(z) &= \exp(1/z), \quad (z \neq 0), \quad z_0 = 0,\\
d)\ f(z) &= z^2(z^2+a^2)^{-3}, \quad (z \neq \pm ia), \quad z_0 = ia, -ia,\\
e)\ f(z) &= (1+z^2+z^4)^{-1}, \quad (z \neq \exp(ri\pi/3), r = 1,2,4,5), \quad z_0 = \exp(i\pi/3).
\end{aligned}$$

2. Find all isolated singularities (also at infinity) and the residues there, for

$$\begin{array}{llll}
a)\ f(z) & = \frac{1}{z^3 - z^5}, & b)\ f(z) & = \sin(z)\sin(1/z),\\
c)\ f(z) & = \frac{e^z}{z^2(z^2+5)}, & d)\ f(z) & = \cot^3(z).
\end{array}$$

3. Using $s(t) = e^{it}$, $t \in [0, 2\pi]$, find

$$\int_s \frac{dz}{z^2 - 2az + 1}, \quad a > 1,$$

and from that

$$\int_0^{2\pi} \frac{dt}{a - \cos(t)}.$$

4. Check the following relations

$$\begin{array}{llll}
a) & \int_0^{2\pi} \left(\cos^4(t) + \sin^4(t)\right) dt = \frac{3\pi}{2}, & b) & \displaystyle\int_0^{2\pi} \left(2\cos^3(t) + 3\cos^2(t)\right) dt = 3\pi,\\
c) & \int_0^{\pi} \frac{dt}{1 + b\cos^2(t)} = \frac{\pi}{\sqrt{b+1}}, \quad b > -1, & d) & \displaystyle\int_0^{\infty} \frac{dx}{1+x^4} = \frac{\pi}{2\sqrt{2}},\\
e) & \int_{-\infty}^{\infty} \frac{(10x)^2}{(x^2+4)^2(x^2+9)^2} dx = \pi \text{ or } \pi/15, & f) & \displaystyle\int_0^{\infty} \frac{\cos(5x)}{x^4+a^4} = \frac{\pi}{2a^3} e^{5a/\sqrt{2}} \sin(5a/\sqrt{2} + \pi/4), \quad a > 0,\\
g) & \int_0^{\infty} \frac{x^2}{(x^2+a^2)^3} dx = \frac{\pi}{16a^3}. & &
\end{array}$$

5. (Hard!!) Show that

$$\int_0^{\infty} \frac{\log(x)}{1+x^2} dx = 0, \qquad \int_0^{\infty} \frac{(\log(x))^2}{1+x^2} = \frac{\pi^3}{8}.$$

Hints: Start out with the second expression. Use a contour which goes a) from $-R$ to $-\epsilon$ along the real axis, then b) along a small half-circle of radius ϵ around $z = 0$, then c) along the real axis from ϵ to R and d) back along a large half-circle of radius R to $-R$. Note that on the negative half-axis, $\log(x) = \log(|x|) + i\pi$. Then you should get a sum of a bunch of terms being equal to the residue at one pole (at $z = i$). Check which terms go to zero and equate the rest. This will give you the result to both of the integrals above in one go. Good luck!