Mathematical Methods.

Problem set 6.
Hand-in 27/10-2008

Problem set 6 is about revision of all the tricks you will need later on to solve ODEs. It's a bit repetitive, but is good training to get into the right state of mind...think of it as going to the gym.

1. **Basic stuff...** Solve the differential equations

$$
\begin{array}{rr}
a)\;\; y' = ky, & b)\;\; y' = -xy, \\
c)\;\; y' = 2\sqrt{1-y^2}/x, & d)\;\; y' = (1+x^2)(1+y^2), \\
e)\;\; xy' = x + y, & f)\;\; x^2y' = y^2 + xy + x^2, \\
g)\, \sinh(x)\cos(y)dx - \cosh(x)\sin(y)dy = 0, & h)\;\; 3ydx + 2xdy = 0,
\end{array}
$$

2. **Initial conditions...** Solve the initial value problems

$$
\begin{array}{c}
a)\;\; y' - y = e^x, \quad y(1) = 0, \\
b)\;\; y' + y = (x+1)^2, \quad y(0) = 0, \\
c)\;\; y' - y\cot(x) = 2x - x^2\cot(x), \quad y(\pi/2) = \pi^2/4 + 1.
\end{array}
$$

3. **Reduction to first order...** Solve

$$a)\;\; xy'' = 2y', \quad b)\;\; xy'' + y' = 0, \quad c)\;\; y'' + (y')^2 = 0.$$

4. **Second order with constant coefficients...** Solve the equations

$$
\begin{array}{rl}
a)\;\; y'' + 4y' + 5y = 0, & b)\;\; y'' + y' - 2y = 0, \\
c)\;\; y'' - 16y = 0, \;\; y(0) = 3, \;\; y(1/4) = 3e, & d)\;\; y'' + 2\alpha y' + (\alpha^2 + \pi^2)y = 0, \;\; y(0) = 3, \;\; y'(0) = -3\alpha.
\end{array}
$$

5. **General second order ODEs...** Solve the equation

$$a)\;\; y'' + 4y = 4\sec(2x), \quad b)\;\; x^2y'' - 2y = 9x^2, \quad c)\;\; 4x^2y'' + 4xy' - y = 12/x,$$

6. **Power series...** Solve

$$a)\;\; xy' = 3y + 3, \quad b)\;\; (1-x^2)y' = 2xy, \quad c)\;\; y'' + y = 2x^2 + x,$$

7. **Frobenius method...** Solve

$$a)\;\; x(1-x)y'' + 2(1-2x)y' - 2y = 0, \quad b)\;\; 16x^2y'' + 3y = 0, \quad c)\;\; 2x^2y'' + xy' - 3y = 0,$$

8. **Bessel functions of the first kind of order ν.** Use the Frobenius method to solve

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0, \quad \nu > 0.$$

9. **The Gamma function** For the Bessel function above, it is conventional to fix $c_0 = 1/(2^\nu \Gamma(\nu+1))$, where $\Gamma(x)$ is the analytic continuation of the factorial to all real (or indeed complex) numbers

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt.$$

Show that

$$\Gamma(x+1) = x\Gamma(x), \qquad \Gamma(x+1) = x!$$

Using this normalisation we denote the two solutions as $J_{\pm\nu}$. Show that they are not linearly independent when n is integer.

10. **Bessel functions of the second kind** (Harder. Have a go!) Take $n = 0$, and find the second solution which is linear independent from J_0. Show that one can choose such a solution to be

$$Y_0(x) = \frac{2}{\pi}\left[J_0(x)\left(\ln(x/2) + \gamma\right) + \sum_1^\infty \frac{(-1)^{m-1} h_m}{2^{2m}(m!)^2} x^{2m}\right],$$

where $h_m = 1 + 1/2 + 1/3 + ... + 1/m$ and the Euler constant is $\gamma = \lim_{m\to\infty}(h_m - \ln(m))$. Show that

$$Y_0(x) = \lim_{\nu\to 0} Y_\nu = \lim_{\nu\to 0} \frac{1}{\sin(\nu\pi)}\left[J_\nu(x)\cos(\nu\pi) - J_{-\nu}(x)\right].$$

Y_ν is the Bessel function of the second kind.