Mathematical Methods.

Problem set 7.
Hand-in 3/11-2008

Problem set 7 is about a set of tools we will need later on:
Orthogonal polynomials, Fourier series, Fourier transform.

1. **Orthogonal polynomials:** a) Show explicitly that the first 5 of the Hermite polynomials

$$He_0 = 1, \quad He_n = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} \left(e^{-x^2/2} \right)$$

   are orthogonal on $]-\infty, \infty[$ with respect to the weight $p(x) = e^{-x^2/2}$. b) Make them orthonormal.

2. **Fourier series:** Find the Fourier series for

$$a) \;\; f(x) = \frac{1}{4}x^2, \quad x \in [-\pi, \pi], \qquad b) \;\; f(x) = |x|, \quad x \in [-\pi, \pi],$$
$$c) \;\; f(x) = x^2, \quad x \in [-\pi/2, \pi/2] \text{ and } \pi^2/4, \quad x \in [\pi/2, 3\pi/2].$$

   d) Use one of these to show that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

3. **Real Fourier transform:** a-c) Find the real Fourier series for the functions above.

4. **The quantum harmonic oscillator and Gaussian integration:** Given the Hamiltonian of the harmonic oscillator

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{m\omega^2}{2}x^2,$$

   we have the time-independent Schrödinger equation

$$\hat{H}\langle x|n\rangle = E_n \langle x|n\rangle,$$

   for some energy eigenstates $\langle x|n\rangle$ with eigenvalues E_n. a) Show that this is a Sturm-Liouville problem, and identify $r(x)$, $p(x)$ and $q(x)$.

   Then we know that we have a set of orthogonal polynomials $\psi_n(x)$ corresponding to the eigenvalues. By writing

$$\langle x|n\rangle = c e^{-ax^2} h_n(x)$$

   b) Show that the functions $h_n(x)$ are the Hermite polynomials. c) Calculate the correlation functions

$$\langle x^m \rangle = \langle n|x^m|n\rangle = \int_{-\infty}^{\infty} x^m \langle n|x\rangle\langle x|n\rangle dx.$$

   for $n, m = 0, 1, 2, 3$.

5. **Fourier transform:** Calculate the Fourier transforms of

$$a)\ \ f(x) = e^{-k|x|}, \quad b)\ \ f(x) = x^n e^{-ax^2}, \quad c)\ \ f(x) = \frac{1}{x^2 + a^2}, \quad d)\ \ f(p_0) = \frac{1}{p_0^2 - \omega_p^2 \pm i\epsilon}$$

$k > 0$, $n = 0, 1, 2$, $a > 0$, $\epsilon > 0$, $\omega_p > 0$. In d) use the inverse Fourier transform with $\int dp_0 e^{ip_0 t}$ (so with t instead of x).