1. Consider the Lagrange density of a complex scalar field

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi$$

where $\phi = \phi_1 + i\phi_2 \in \mathrm{C}$.

a) Show that you obtain the correct equations of motion if you consider $\phi$ and $\phi^*$ to be formally independent functions, i.e.

$$\frac{\partial \mathcal{L}}{\partial \phi^*} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} = 0$$

(+ the hermitean conjugate) gives identical equations of motion to the ones obtained using independent components $\phi_1$ and $\phi_2$.

b) Show that $\mathcal{L}$ is invariant under transformations $\phi \to e^{i\theta}\phi$, $\phi^* \to e^{-i\theta}\phi$, $\theta$ constant. Calculate the corresponding Noether current (consider infinitesimal transformation).

2. Show that $T^{00} = \mathcal{H}$.

3. Show that

$$e^{\hat{A}} \hat{B} e^{-\hat{A}} = \hat{B} + [\hat{A}, \hat{B}] + \frac{1}{2!}[\hat{A}, [\hat{A}, \hat{B}]] + \frac{1}{3!}[\hat{A}, [\hat{A}, [\hat{A}, \hat{B}]]] + \dots$$

Use this result to directly solve the time evolution of the operator $\hat{x}_H(t)$ for harmonic oscillator: thus, defining

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2 \,,$$

evaluate

$$\hat{x}_H(t) \equiv e^{i\hat{H}t} \hat{x} e^{-i\hat{H}t}.$$