1. Let us consider the harmonic oscillator as discussed in the lecture notes. Defining

$$\hat{a} = \sqrt{\frac{m\omega}{2}}\hat{x} + \frac{i}{\sqrt{2m\omega}}\hat{p} \qquad \hat{a}^\dagger = \sqrt{\frac{m\omega}{2}}\hat{x} - \frac{i}{\sqrt{2m\omega}}\hat{p}$$

and using

$$[\hat{x}, \hat{p}] = i, \qquad [\hat{x}, \hat{x}] = [\hat{p}, \hat{p}] = 0$$

show that

a) $[\hat{a}, \hat{a}] = [\hat{a}^\dagger, \hat{a}^\dagger] = 0$, $[\hat{a}, \hat{a}^\dagger] = 1$,

b) $\hat{H} = \omega(\hat{a}^\dagger \hat{a} + \frac{1}{2})$

c) $\hat{N} \equiv \hat{a}^\dagger \hat{a}$ is hermitean (and has real eigenvalues)

d) If we define states $|n\rangle$ so that $\hat{N}|n\rangle \equiv n|n\rangle$ and $\langle n|n\rangle = 1$, show that $\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$ and $\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$.

2. Consider the Fock-space of scalar field theory. Thus, define 2-particle states through $|\vec{k}_1, \vec{k}_2\rangle \equiv \hat{a}^\dagger_{\vec{k}_1} \hat{a}^\dagger_{\vec{k}_2}|0\rangle$.

a) Show that $|\vec{k}_1, \vec{k}_2\rangle = |\vec{k}_2, \vec{k}_1\rangle$ (states are symmetric wrt. permutations, i.e. bosons)

b) What is $\langle \vec{q}_1, \vec{q}_2|\vec{k}_1, \vec{k}_2\rangle$?

3. Defining

$$\hat{N} \equiv \int d^3\vec{p}\, \hat{a}^\dagger_{\vec{p}} \hat{a}_{\vec{p}}, \qquad :\hat{H}: \equiv \int d^3\vec{p}\, E_{\vec{p}}\, \hat{a}^\dagger_{\vec{p}} \hat{a}_{\vec{p}},$$

calculate $\hat{N}|\vec{k}_1, \vec{k}_2, \ldots \vec{k}_n\rangle$ and $:\hat{H}:|\vec{k}_1, \vec{k}_2, \ldots \vec{k}_n\rangle$.