1. We define the Fourier transform of a propagator $G(x,y) = G(x-y)$ through

$$G(p) = \int d^4x \, e^{ip\cdot x} G(x) \, , \qquad G(x) = \int \frac{d^4p}{(2\pi)^4} \, e^{-ip\cdot x} G(p)$$

   a) What is $G_F(p)$ for the scalar field?

   b) Show that

$$\rho(p) = \frac{\pi}{2p^0} [\delta(p^0 - E_{\vec{p}}) + \delta(p^0 + E_{\vec{p}})]$$

   This is called the spectral density (for the free scalar field).

   c) Show that $\rho(p) = \mathrm{Im}\, G_R(p)$.
   Hint: show that the contour in (1.46) can be defined by adding a small imaginary part to p^0. Then you can use the very useful relation

$$\frac{1}{x + i\epsilon} = \mathrm{P}\frac{1}{x} - i\pi\delta(x),$$

   where P is the Cauchy principal value.
   Note: on page 35–36 the retarded propagator should be multiplied by i in order to obtain (1.34). While the definition on pg. 35 is more symmetric wrt. other propagators, (1.34) is perhaps more standard and the above result is for that convention.

2. Show that the solution of the non-homogeneous equation ($J(x)$: source term)

$$(\partial^\mu \partial_\mu + m^2)\psi(x) = J(x)$$

   can be written as

$$\psi(x) = \psi_0(x) + i \int d^4y G(x,y) J(y)$$

   where ψ_0 is a solution for the homogeneous equation and G is a Green function. Convince yourself that if G is the retarded propagator, $J(y)$ can affect $\psi(x)$ only if y is in the past light cone of point x (i.e. $y^0 \leq x^0$, $(x-y)^2 \geq 0$).