Quantum Field Theory Spring 2008 Problem set 4, session 13.3, return by 18.3.

1. Show that
$$\frac{\delta}{i\delta J(x_1)}\frac{\delta}{i\delta J(x_2)}\frac{\delta}{i\delta J(x_3)}Z[J]\Big|_{J=0} = 0.$$

2. *Harmonic oscillator in the path integral formalism:*
The harmonic oscillator action is
$$S[x] = \int dt\left(\frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2x^2\right),$$
where $x(t)$ is the trajectory of the oscillator.

   a) Let us write this in form (after partial integration)
$$S = -\frac{1}{2}x^T Ax \equiv -\frac{1}{2}\int dt\, dt' x(t)A(t,t')x(t)\,.$$
What is the "matrix" $A(t,t')$? (Note: you will rather get an expression $\int dt x(t)Ax(t)$, but in order to make A more matrix-like 2-index object you can multiply it with $\delta(t-t')$.)

   b) Let us define the generating functional
$$Z[J] = \int[\prod_t dx(t)]\exp\left[-i\frac{1}{2}x^T Ax + iJ^T x\right]$$
Show that this is
$$Z[J] = Z[0]\exp\left[i\frac{1}{2}J^T A^{-1}J\right].$$
What is the expression for $Z[0]$ (no need to evaluate it)?

   c) A^{-1} obeys the equation $A\,A^{-1} = 1$ (naturally!). Write down this equation in explicit form using the expression for A obtained in a). From this equation we can observe that A^{-1} is a Green function for the differential operator A. We denote here $G(t,t') = A^{-1}(t,t')$.

   d) Taking a Fourier transformation of the equation above, and using
$$G(t,t') = \int\frac{dp}{2\pi}G(p)e^{-ip(t-t')}$$
show that
$$G(p) = \frac{1}{m(\omega^2-p^2)}.$$

e) The pole in the integrand in the Fourier transformation in d) requires regularization which would ensure convergence. Thus, let us modify

$$\omega^2 - p^2 \to \omega^2 - p^2 - i\epsilon, .$$

Evaluating the Fourier transformation in d) (by contour integration) show that

$$G(t,t') = i\frac{e^{i\omega|t-t'|}}{2m\omega}.$$

$G(t,t')$ is the analogue of the Feynman propagator for the harmonic oscillator. It gives us the correlation between the position (x) of the oscillator at time t and t' in ground state, i.e.

$$\langle\psi_0|T\hat{x}(t)\hat{x}(t')|\psi_0\rangle = -iG(t,t')$$

where $|\psi_0\rangle$ is the ground state wave function.