(1)

Problems set 1

1) $\phi = \phi_1 + i\phi_2$

$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi$ free complex scalar

a) Euler-Lagrange eq

$$\frac{\delta \mathcal{L}}{\delta \phi^*} - \partial_\mu \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi^*)} = 0$$

$$= -m^2 \phi - \partial_\mu \partial^\mu \phi^* = 0$$

or $[\partial_t^2 - \partial_x^2 + m^2](\phi_1 + i\phi_2) = 0$

In terms of ϕ_1 & ϕ_2

$$\mathcal{L} = \partial_\mu \phi_1 \partial^\mu \phi_1 - m^2 \phi_1^2 + \partial_\mu \phi_2 \partial^\mu \phi_2 - m^2 \phi_2^2$$

$$\frac{\partial \mathcal{L}}{\delta \phi_1} - \partial_\mu \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi_1)} = 0$$

same + (ϕ_2, ϕ^*, etc)

$$= -m^2 \cdot 2\phi_1 - \partial_\mu 2\partial^\mu \phi_1 = 0$$

or $[\partial_t^2 - \partial_x^2 + m^2]\phi_1 = 0$

b) Invariance under $\phi \to e^{i\theta}\phi$, $\phi^* \to e^{-i\theta}\phi$

θ constant

$$\mathcal{L}' = \partial_\mu e^{-i\theta} \phi^* \partial^\mu e^{i\theta} \phi - m^2 e^{-i\theta} \phi^* e^{i\theta} \phi$$

$$= e^{-i\theta} e^{i\theta} \mathcal{L} = \mathcal{L} \quad \text{because } \theta \text{ constant}$$

Noether current: infintesimal & assume θ not constant

$$\partial_\mu (1 - i\theta)\phi^* \partial^\mu (1 + i\theta)\phi - m^2 (1 - i\theta)\phi^* (1 + i\theta)\phi$$

$$= \mathcal{L} - i\theta \partial_\mu \phi^* \partial^\mu \phi + i\theta \partial_\mu \phi^* \partial^\mu \phi - m^2 (i\theta - i\theta)\phi^* \phi$$

$$- i\partial_\mu \theta [\phi^* \partial^\mu \phi - \partial^\mu \phi^* \phi] + \mathcal{O}(\theta^2)$$

$$\Rightarrow j^\mu_{\text{noether}} = -i[\phi^* \partial^\mu \phi - \partial^\mu \phi^* \phi]$$

②

Problems set 1

2) $T^{00} = \mathcal{H}$?

$$T^{\mu\nu} = \frac{\delta \mathcal{L}}{\delta(\partial_\mu \varphi)} \partial^\nu \varphi - g^{\mu\nu} \mathcal{L}$$

$$T^{00} = \frac{\partial \mathcal{L}}{\delta(\partial_0 \varphi)} \partial^0 \varphi - g^{00} \mathcal{L} \qquad (g^{00} = 1)$$

$$= (\partial_0 \varphi)^2 - \mathcal{L} = \Pi \dot\varphi - \mathcal{L} = \mathcal{H}$$

3) $$e^A B e^{-A} = \left(1 + A + \frac{A^2}{2} \cdots\right) B \left(1 - A + \frac{A^2}{2} - \cdots\right)$$

$$= B + AB - BA + \frac{A^2}{2} B + B\frac{A^2}{2} - ABA \ldots$$

$$= B + [A, B] + \frac{1}{2}[A,[A,B]] \longrightarrow \text{Proof below}$$

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m^2 \omega^2}{2} \hat{x}^2 \quad \text{(Harmonic oscillator)}$$

$$\hat{x}_H(t) = e^{i\hat{H}t} \hat{x} e^{-i\hat{H}t} \qquad (A = i\hat{H}t)$$

$$[i\hat{H}t, \hat{x}] \qquad ([\hat{x}, \hat{p}] = i)$$

$$= \frac{it}{2m} [\hat{p}^2, \hat{x}] = \frac{it}{2m} 2p(-i) = \frac{t\hat{p}}{m}$$

$$[i\hat{H}t, \hat{p}]$$

$$= \frac{itm\omega^2}{2} [\hat{x}^2, \hat{p}] = -tm\omega^2 \hat{x}$$

$$\hat{x}_H(t) = \hat{x} + \frac{t\hat{p}}{m} + \frac{1}{2!} \frac{t}{m} (-tm\omega^2) \hat{x} + \frac{1}{3!} -t^2\omega^2 \frac{t}{m} \hat{p}$$

$$+ \frac{1}{4!} (-t^2\omega^2)^2 \hat{x} + \ldots$$

$$= \hat{x}\left(1 \frac{(-t^2\omega^2)}{2} + \frac{(-t^2\omega^2)^2}{4!}\right) = \hat{x} \cos \omega t \qquad \boxed{x(0) = x,\ x'(0) = \frac{p}{m}}$$

$$\frac{t\hat{p}}{m}\left(1 - \frac{t^2\omega^2}{3!} + \frac{(t^2\omega^2)^2}{5!} - \ldots\right) \qquad \left(\boxed{\hat{x}(t) = \hat{x}_0 \cos\omega t + \frac{\hat{p}_0}{\omega m} \sin\omega t}\right.$$

$$= \frac{\hat{p}}{\omega m} \sin \omega t$$

Problem set 1,

3) $e^{A} B e^{-A} =$

$$\sum_{n=0}^{\infty} \frac{A^n}{n!} \; B \; \sum_{m=0}^{\infty} \frac{(-A)^m}{m!} = \sum_{n=0}^{\infty}\sum_{i}^{m} \frac{A^{n-i}}{(n-i)!} B \frac{A^i}{i!}(-1)^i$$

$$[A,B] = AB - BA$$

$$[A,[A,B]] = AAB - 2ABA + BAA = A^2B - 2ABA + BA^2$$

$$\frac{1}{3!}[A,[A,[A,B]]] = (A^3B - 3A^2BA + 3ABA^2 - BA^3)\,1/6$$

$$= \sum_{i=0,n} \frac{A^{n-i}\; B\; A^i (-1)^i}{(n-i)!\; i!}$$

$n=3$: $\frac{1}{6}, \frac{1}{2}, \frac{1}{2}, \frac{1}{6} \sim \frac{1}{6}(1,3,3,1)$

$n=4$: $\frac{1}{24}, \frac{1}{6}, \frac{1}{4}, \frac{1}{6}, \frac{1}{24} \sim \frac{1}{24}(1,4,6,4,1)$

Pascal's triangle

kall that

$$e^{A} B e^{-A} = \sum_{n=0}^{\infty}\sum_{i=0}^{n} \frac{A^{n-i}\; B\; A^i (-1)^i}{(n-i)!\; i!}$$

$$= B + [A,B] + \frac{1}{2!}[A,[A,B]] + \frac{1}{3!}[A,[A,[A,B]]] \ldots$$

Problems set 2

1) Harmonic oscillator

$$\hat{a} = \sqrt{\frac{m\omega}{2}}\,\hat{x} + \frac{i}{\sqrt{2m\omega}}\,\hat{p}$$

$$\hat{a}^\dagger = \sqrt{\frac{m\omega}{2}}\,\hat{x} - \frac{i}{\sqrt{2m\omega}}\,\hat{p}$$

$$[\hat{x},\hat{p}] = i \qquad [\hat{x},\hat{x}] = [\hat{p},\hat{p}] = 0$$

a) $[a,a] = [a^\dagger,a^\dagger] = 0$ Trivial

$$[\hat{a},\hat{a}^\dagger] = -\sqrt{\frac{m\omega}{2}}\frac{i}{\sqrt{2m\omega}}\left([\hat{x},\hat{p}] - [\hat{p},\hat{x}]\right)$$

$$= -\frac{i}{2}(i+i) = \underline{1}$$

b) : $H \overset{?}{=} \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{x}^2$

$$\omega(a^\dagger a + 1/2) =$$

$$\omega\left(\left(\sqrt{\frac{m\omega}{2}}\right)^2\hat{x}^2 + \left(\frac{1}{\sqrt{2m\omega}}\right)^2\hat{p}^2 + 1/2 + \sqrt{\frac{m\omega}{2}}\frac{i}{\sqrt{2m\omega}}[x,p]\right)$$

$$= \frac{m\omega^2}{2}\hat{x}^2 + \frac{\hat{p}^2}{2m} + \omega(1/2 - 1/2) = H$$

c) : $\hat{N} = a^\dagger a$ is hermitian

$(a^\dagger a)^\dagger = a^\dagger a$!! since $a^\dagger = (a)^\dagger$, $x^\dagger = x$, $p^\dagger = p$.

d) define $|n\rangle$, $\hat{N}|n\rangle = n|n\rangle$, normalized $\langle n|n\rangle = 1$

Oops ~~$a^\dagger|n\rangle = n a^\dagger(a^\dagger a)|n\rangle = n\, a^\dagger(aa^\dagger+1)|n\rangle \quad (a^\dagger a = aa^\dagger + 1)$~~

$$(a^\dagger a)a^\dagger|n\rangle = a^\dagger(a^\dagger a + 1)|n\rangle = a^\dagger(n+1)|n\rangle = (n+1)a^\dagger|n\rangle$$

$$\Rightarrow \hat{N}(a^\dagger|n\rangle) = (n+1)(a^\dagger|n\rangle) \Rightarrow a^\dagger|n\rangle = c\,|n+1\rangle,\ c \in \mathbb{C}$$

$$\langle n+1|c^* c|n+1\rangle = \langle n|a a^\dagger|n\rangle = \langle n|\hat{N}+1|n\rangle = n+1$$

$$\Rightarrow |c|^2 = n+1 \Rightarrow a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

Problem set 2

1.d cont.

$$\hat{N} a|n\rangle = a^\dagger a\, a|n\rangle = (aa^\dagger - 1)\, a|n\rangle$$
$$= a\,(a^\dagger a - 1)|n\rangle = a(n-1)|n\rangle$$
$$\Rightarrow a|n\rangle = c\,|n-1\rangle \quad , \quad c \in \mathbb{C}$$
$$\langle n-1|\,|c|^2\,|n-1\rangle = \langle n|a^\dagger a|n\rangle = \langle n|n|n\rangle = n$$
$$\Rightarrow |c|^2 = n \quad , \quad c = \sqrt{n} \Rightarrow a|n\rangle = \sqrt{n}\,|n-1\rangle$$

2 Define multiparticle states

$$|\vec{k}_1, \vec{k}_2\rangle = a^\dagger_{k_1} a^\dagger_{k_2}|0\rangle \ldots$$

Commute

a) $|\vec{k}_1, \vec{k}_2\rangle = a^\dagger_{k_1} a^\dagger_{k_2}|0\rangle = a^\dagger_{k_2} a^\dagger_{k_1}|0\rangle = |\vec{k}_1, \vec{k}_2\rangle$

b) $\langle \vec{q}_1, \vec{q}_2 | \vec{k}_1, \vec{k}_2\rangle$

$$= \langle 0|\hat{a}_{q_1}\hat{a}_{q_2}\hat{a}^\dagger_{k_1}\hat{a}^\dagger_{k_2}|0\rangle \quad \left([a_k, a^\dagger_q] = \delta_{kq} \right)$$
$$= \langle 0|\hat{a}_{q_1}(a^\dagger_{k_1} a_{q_2} + \delta_{k_1 q_2})\, a^\dagger_{k_2}|0\rangle$$
$$= \langle 0|(a^\dagger_{k_1} a_{q_1} + \delta_{k_1 q_1})\, a_{q_2} a^\dagger_{k_2} + (a^\dagger_{k_2} a_{q_1} + \delta_{q_1 k_2})\,\delta_{k_1 q_2}|0\rangle$$
$$= \langle 0|(a^\dagger_{k_1} a_{q_1} + \delta_{k_1 q_1})(a^\dagger_{k_2} a_{q_2} + \delta_{q_2 k_2}) + (a^\dagger_{k_1} a_{q_1} + \delta_{q_1 k_2})\,\delta_{k_1 q_2}|0\rangle$$
$$= \langle 0|\,\delta_{k_1 q_1}\delta_{q_2 k_2} + \delta_{q_1 k_2}\delta_{q_2 k_1}|0\rangle = \underline{\underline{\delta_{k_1 q_1}\delta_{q_2 k_2} + \delta_{q_1 k_2}\delta_{q_2 k_1}}}$$

3

$$\hat{N} = \int d^3p\, a^\dagger_p a_p \qquad H = \int d^3_p\, E_p\, a^\dagger_p a_p$$

$$N|k_1 \ldots k_n\rangle = \int d^3p\,(\delta_{k_1 p} + \delta_{k_2 p} + \ldots) = n$$
$$H|\qquad\rangle = \int d^3p\, E_p\,(\delta_{k_1 p} + \delta_{k_2 p} + \ldots) = E_1 + E_2 + \ldots$$
$$(E_n \equiv E_{k_n})$$

Problem set 3   1.a $G_F(p) = \frac{i}{p^2 - m^2 + i\epsilon}$

1.b

show $\rho(p) = \frac{\pi}{2p_0}\{\delta(p^0 - E_p) + \delta(p^0 + E_p)\}$

$$\rho(x,y) = \langle \tfrac{1}{2}[\phi_x, \phi_y]\rangle = \int \frac{d^3p}{(2\pi)^3}\, G(\bar{p}, t)$$

$$G(\bar{p},t) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p} e^{-ip(x-y)} \frac{1}{2}\left(e^{-iE_p(x^0-y^0)} - e^{iE_p(x^0-y^0)}\right)$$

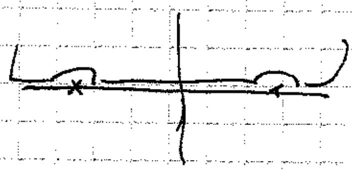
Fourier transform w.r.t. $x_0 - y^0$ $\int d(x^0-y^0) \frac{e^{+ip^0(x^0-y^0)}}{2E_p} \frac{1}{2}\left[e^{-iE_p(x^0-y^0)} - e^{iE_p(x^0-y^0)}\right]$

$$= \frac{1}{4E_p} 2\pi\left[-\delta(p^0 + E_p) + \delta(p^0 + E_p)\right]$$

$$= \frac{\pi}{2p^0}\left\{\delta(p^0 - E_p) + \delta(p^0 + E_p)\right\}$$

c)   convention

$$G_R(x) = \int \frac{d^4p}{(2\pi)^4}\frac{i \cdot i}{p^2 - m^2} e^{-ip(x-y)}$$



Use $\int \frac{d^4p}{(2\pi)^4}\frac{-1}{-\bar{p}^2 - m^2 + (p_0 + i\epsilon)^2} e^{-ip(x-y)}$

Same poles $(p_0 + i\epsilon) = \pm\sqrt{p^2 + m^2}$

Use $\frac{-1}{p_0^2 + i2\epsilon p_0 - (\bar{p}^2 + m^2)} = -\left[P\left(\frac{1}{x}\right) - i\pi\,\delta(p_0^2 - (\bar{p}^2 + m^2))\right]$

$$\mathrm{Im}\, G_R^{(p)} = -\pi\,\delta(p_0^2 - (\bar{p}^2 + m^2))$$

Use $\delta(f(x)) = \frac{1}{|f'(x)|}\delta(x) \Sigma$ at all zero's of $f(x)$

$f(p_0) = p_0^2 - (\bar{p}^2 + m^2)$ $f'(p_0) = 2p_0$ $\Rightarrow + \frac{\pi}{2|p_0|}\left(\delta(p_0 + E) + \delta(p_0 - E)\right)$

$\Rightarrow \frac{1}{2|p_0|}\left[\delta(p_0 + \sqrt{p^2 + m^2}) + \delta(p_0 - \sqrt{p^2 + m^2})\right]$ $\Rightarrow \underline{\rho(p) = \mathrm{Im}\, G_R(p)}$

Problem set 3

2)
$$(\partial^\mu \partial_\mu + m^2)\,\psi(x) = J(x)$$

solution ? $\psi(x) = \psi_0(x) + i \int d^4y\, G_{xy} J_y$, ψ_0 hom. solution

Test: $(\partial^\mu \partial_\mu + m^2)\left[\psi_0 + i \int d^4y\, G_{xy} J_y\right]$

$$= 0 + i \int d^4y\, {}_z(\partial_\mu \partial^\mu + m^2)_x\, G_{xy} J_y$$

G is Green's function, &

${}_z(\partial_\mu \partial^\mu + m^2)_x\, G_{xy} = -i\delta_{zy}$ by definition

$$= i \int d^4y\, (\quad)_{zx}\, G_{xy} J_y = +\int d^4y\, \delta_{zy} J_y = +J_z$$

If G is G_{Retarded}, then

$G_{xy} = 0$ outside past lightcone $y^0 \leq x^0$ $(x-y)^2 \geq 0$

$\Rightarrow$ $G_{xy} J_y$ is also 0 outside lightcone of x

and so any component of $J(y)$ outside the lightcone has no effect on $\psi(x)$.

Problems set 4

1) $$\frac{\delta}{i\delta J_{x_1}} \frac{\delta}{i\delta J_{x_2}} \frac{\delta}{i\delta J_{x_3}} Z[J]\Big|_{J=0} = 0$$

for free scalar field ?

$$\frac{Z[J]}{Z[0]} = e^{-1/2 \int_{xy} J_x G_{xy} J_y}$$

$$\frac{\delta}{i\delta J_{x_3}} \Rightarrow \tfrac{+i}{2}\int_x G_{x x_3} J_x + \tfrac{i}{2}\int_x G_{x_3 x} J_x \, e^{-1/2\, J_x G_{xy} J_y}$$

$$- i \int_x G_{x_3 x} J_x \; e^{-1/2\, J_x G_{xy} J_y}$$

$$\frac{\delta}{i\delta J_{x_2}} \Rightarrow -G_{x_3 x_2} e^{(\;)} + (-i)^2 \int_x G_{x_3 x} J_x \int_{x'} G_{x_2 x'} J_{x'} e^{(\;)}$$

$$\frac{\delta}{i\delta J_{x_1}} \Rightarrow -G_{x_3 x_2} e^{(\;)} \left(-i \int_x G_{x_1 x} J_x\right)$$
$$+ (-i)^3 \int_x G_{x_3 x} J_x \int_{x'} G_{x_2 x'} J_{x'} \int_{x''} G_{x_1 x''} J_{x''}\, e^{(\;)}$$
$$+ i \left[\int_x G_{x_3 x} J_x G_{x_2 x_1} + \int_x G_{x_2 x} J_x G_{x_3 x_1}\right.$$

$$J = 0 \Rightarrow 0$$

2) Harm. osc: $S[x] = \int dt \left(\frac{m\dot{x}^2}{2} - \frac{m\omega^2 x^2}{2}\right)$

a) $= -\frac{1}{2}\int dt\, dt'\, x(t)\, A(t,t')\, x(t')$

For $A(t,t') = m\,(\partial_t^{\,2} + \omega^2)\,\delta(t,t')$

b) $Z[J] = \int \prod_t dx \exp\left[-i\,\frac{x^T A x}{2} + i J^T x\right]$

Problem set 4

2.b $-i\frac{1}{2}x^T A x + iJx$

$$= -\frac{i}{2}\left(x - \overset{A^{-1}}{J}\right)^T A\left(x - A^{-1}J\right) \qquad \left(A^{-1^T} = A^{-1}\right)$$

(Complete the square)

$$= -\frac{i}{2}x^T A x - \frac{i}{2}J^T(A^{-1})^T A A^{-1} J + \frac{i}{2}J^T(A^{-1})^T A x + \frac{i}{2}x^T A A^{-1} J$$

$$= -\frac{i}{2}x^T A x + iJ^T x - \frac{i}{2}J^T A^{-1} J$$

$$\Rightarrow Z[J] = Z[0]\exp\left[\frac{i}{2}J^T A^{-1} J\right]$$

With $Z[0] = \int \prod_t dx(t)\, e^{-\frac{i}{2}(x - A^{-1}J)^T A (x - A^{-1}J)}$

$dy(t)$, $y(t) = x(t) - A^{-1}J(t)$

c) $AA^{-1} = 1 : \int_{t'} m\left(\frac{d^2}{dt^2} + \omega^2\right)\delta(t-t')\, A^{-1}_{t',t''} = \delta(t-t'')$

$G(t,t') = A^{-1}(t,t')$

d) $\int_{t'} m\left(\frac{d^2}{dt^2} + \omega^2\right)\delta(t-t')\int \frac{dp}{2\pi} G(p)\, e^{-ip(t'-t'')}$

$$= \int_{t'}\int \frac{dp}{2\pi}\delta(t-t')\frac{m(\omega^2-p^2)}{m(\omega^2-p^2)}e^{-ip(t'-t'')}$$

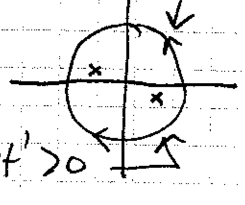
$$= \int \frac{dp}{2\pi} e^{-ip(t-t'')} = \underline{\underline{\delta(t-t'')}}$$

$\frac{1}{m(\omega^2-p^2)}$ is Green's function of $A(t,t')$

e) $\omega^2 - p^2 \to \omega^2 - p^2 - i\varepsilon$

$\int \frac{dp}{2\pi}\frac{1}{m(\omega^2-p^2-i\varepsilon)}e^{-ip(t-t')}$ by contour integration

$t-t'<0$



$t-t'>0$

Poles at $p = \pm\sqrt{\omega^2 - i\varepsilon}$: $\frac{i}{2m\omega}e^{-i\omega|t-t'|}$

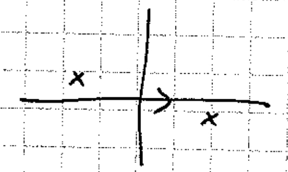
Problemset 4,

2.e) cont.

$$\int \frac{dp}{2\pi} \frac{1}{m(\omega^2-p^2-i\epsilon)} e^{-ip(t-t')}$$

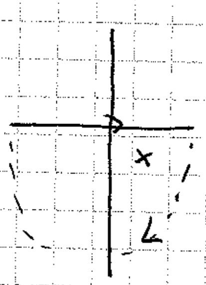
poles: $p^2 = \omega^2 - i\epsilon$

$p = \pm\sqrt{\omega^2-i\epsilon}$



$e^{-ip(t-t')}$: $t-t'>0$

need to close below: $-ip = -|p|$



$$\frac{2\pi i}{2\pi} \frac{p-\sqrt{\omega^2-i\epsilon}}{m(\omega^2-p^2-i\epsilon)} e^{-i\sqrt{\omega^2-i\epsilon}(t-t')}$$

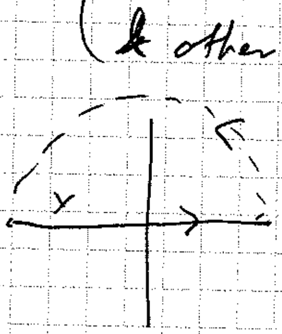
$$= \frac{2\pi i}{2\pi m} \frac{1}{-(p+\sqrt{\omega^2-i\epsilon})} e^{-i\sqrt{\omega^2-i\epsilon}(t-t')}$$ (−) counterclockwise

$$= \frac{2\pi i}{2\pi m} \frac{1}{p+\omega} e^{-i\omega(t-t')} = \frac{2\pi i}{2\pi 2m\omega} e^{-i\omega(t-t')}$$

$$= \frac{i}{2m\omega} e^{-i\omega(t-t')}$$

($t-t'<0$
other way)

$$\boxed{\frac{i}{2m\omega} e^{-i\omega|t-t'|} = G(t,t')}$$



Problem set 5

2) $$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{m^2}{2}\phi^2 - \frac{1}{3!}g\,\phi^3$$

Free Feynman propagator

$$G_F(p) = \frac{i}{p^2-m^2+i\epsilon}$$

$$Z[J] = e^{iS_I[\delta/i\delta J]}\,Z_0[J]$$

$$= \left[1 - \frac{ig}{3!}\left(\frac{\delta}{i\delta J(x)}\right)^3\right]\underbrace{e^{-\frac{1}{2}J_x G_{xy} J_y}}_{Z_0[J]}$$

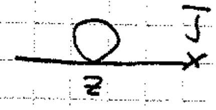
To order g $Z[J]$

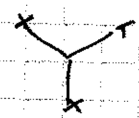
$$\frac{\delta}{i\delta J_z}Z_0[J] = \frac{-2}{2i}G_{zy}J_y Z_0[J] = iG_{zy}J_y Z_0[J]$$

$$(\quad)^2 = (iG_{zy}J_y)^2 Z_0 + G_{zz}Z_0$$

$$(\quad)^3 = (iG_{zy}J_y)^3 Z_0 + \frac{2}{i}\, iG_{zy}J_y\, iG_{zz}Z_0$$

$$Z[J] = 1 - \frac{ig}{6}\left[2iG_{zz}G_{zy}J_y Z_0 - i(G_{zy}J_y)^3 Z_0\right]$$

$= 1 + \frac{g}{3}$ 

$- \frac{g}{6}$ 

$$Z[0] = 1$$

Problem set 5

2)

$$\langle \phi_x \phi_y \rangle = G^{2} = \frac{\delta}{i\delta J_x}\frac{\delta}{i\delta J_y}\frac{Z[J]}{Z[0]}\Big|_{J=0}$$

$$= -\frac{\delta}{\delta J_x}\frac{\delta}{\delta J_y}\left[1 + \frac{g}{3} G_{zz} G_{zy} J_y - \frac{g}{6}\left(G_{zy}J_y\right)^3\right] Z_0 \Big|_{J=0}$$

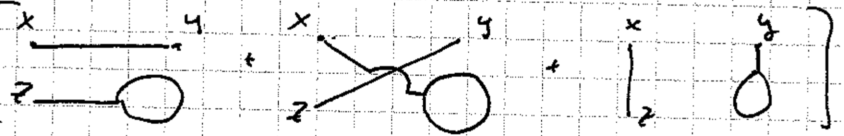
$$= + G_{xy} + O \ldots$$

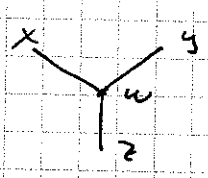
$$\langle \phi_x \phi_y \phi_z \rangle = G^{(3)} = \frac{\delta}{i\delta J_x}\frac{\delta}{i\delta J_y}\frac{\delta}{i\delta J_z}\frac{Z[J]}{Z[0]}\Big|_{J=0}$$

$$= i\frac{\delta}{\delta J_x}\frac{\delta}{\delta J_y}\frac{\delta}{\delta J_z}\left[1 + \frac{g}{3} G_{ww} G_{w\xi} J_\xi - \frac{g}{6}\left(G_{w\xi}J_\xi\right)^3\right] Z_0 \Big|_{J=0}$$

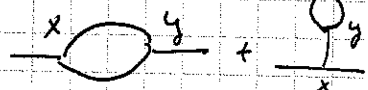
$$= i\frac{g}{3}\left[-G_{ww}G_{wz}G_{xy} - G_{ww}G_{wx}G_{yz} - G_{ww}G_{wy}G_{xz}\right]$$

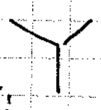
$$-i g\, G_{wx}G_{wy}G_{wz}$$

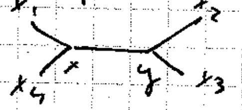
$= -i\frac{g}{3}\Big[$  $\Big]$ not connected!

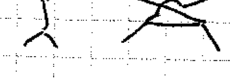
$-izg$  Connected!

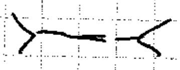
Feynman rules: $-ig$ ≺ Vertex — propagator.

2-point function $G(g^2)$:  $= \frac{1}{2}(-ig)^2 \begin{cases} G_{xy}^2 \\ G_{xy}G_{yy} \end{cases}$

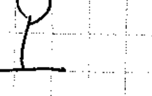
3-point function :  $(-ig)$

4-point function  $= (-ig)^2 G_{xy}$

+ crossing 

Symmetry Factors:  : $\frac{1}{2!}\left(\frac{1}{3!}\right)^2 \cdot 6 \cdot 2 \cdot 3 \cdot 2 = 1$

 $\frac{1}{2!}\left(\frac{1}{3!}\right)^2 \cdot 6 \cdot 3 \cdot 2 = \frac{1}{2}$

 $\frac{1}{2!}\left(\frac{1}{3!}\right)^2 \cdot 6 \cdot 2 \cdot 3 = \frac{1}{2}$

Problemset 5

3.)

$$\frac{1}{ab} = \int_0^1 \frac{dx}{[ax + b(1-x)]^2} \quad ? \quad a,b>0$$

$$\int_0^1 \frac{dx}{[x(a-b)+b]^2} \qquad y = x(a-b)+b$$

$$dy = dx(a-b)$$

$$\frac{1}{(a-b)}\int_b^{a} \frac{dy}{y^2} = \frac{-1}{a-b}\left[\frac{1}{y}\right]_b^{a} = \frac{-1}{a-b}\left(\frac{1}{a}-\frac{1}{b}\right)$$

$$= \frac{1}{ab}$$

Problem set 6

1.)
$$I(q^2, m^2) = \int \frac{d^d p}{(2\pi)^d} \frac{1}{p^2+m^2} \frac{1}{(p+q)^2+m^2}$$

Show independence of $\frac{\hat{q}}{|q|}$, external momentum direction

$q \to q' = Aq$, A rotation matrix, $A^T A = 1$

$$\frac{1}{(p+q')^2+m^2} = \frac{1}{A^T A^*(A^{-1}p+q)^2+m^2}$$

$$I(q^2, m^2) = \int \frac{d^d p \det A^{-1}}{(2\pi)^d} \frac{1}{(p^2+m^2)} \frac{1}{(A^T p+q)^2+m^2}$$

Since $A \Rightarrow \det A = 1$, and integrate over all p,

independent of $A \Rightarrow q'$, direction of q

2.)
$$\int \frac{d^d p}{(2\pi)^d} \frac{1}{p^2+m^2} \frac{1}{(p+q)^2+m^2} = \int \frac{1}{ab}$$

$$= \int \frac{d^d p}{(2\pi)^d} \int_0^1 \frac{d\alpha}{\left\{(p^2+m^2)\alpha + ((p+q)^2+m^2)(1-\alpha)\right\}^2}$$

$$p^2\alpha + m^2\alpha + (p^2+2p\cdot q+q^2+m^2(1-\alpha))$$

$$= p^2+q^2+m^2+2p\cdot q - \alpha q^2$$

Write $p' = p+(1-\alpha)q \Rightarrow p = p'-(1-\alpha)q$

$$\cancel{q^2+m^2-\alpha q^2+2p'q-2(1-\alpha)q^2-2p'(1-\alpha)q+p'^2+(1-\alpha)^2q^2}$$

$$= \cancel{m^2+q^2(1-\alpha-2+2\alpha+1-2\alpha+\alpha^2)+p'^2+2\alpha p'q}$$

$$= \cancel{m^2+\alpha^2q^2+p'^2+2\alpha p'q} = p'^2+m^2+\alpha(1-\alpha)q^2$$

$$\alpha(p'-(1-\alpha)q)^2 + (1-\alpha)(p'+\alpha q)^2$$

$$\alpha p'^2 + \alpha(1-\alpha)^2 q^2 + (1-\alpha)p'^2 + \alpha^2(1-\alpha)q^2$$

$$-2\alpha(1-\alpha)p'\cdot q + 2\alpha(1-\alpha)p'\cdot q$$

$$= p'^2 + \alpha(1-\alpha)q^2$$

Problem set 6, 1

$$\int_0^1 d\alpha \int \frac{d^d p'}{(2\pi)^d} \frac{1}{\left[p'^2 + m^2 + \alpha(1-\alpha) q^2\right]^2}$$ (Note, independent of direction of q. only q^2)

$dp' = dp$:

Use general eq. for $\frac{1}{\left[p'^2 + M^2\right]^2}$, $M^2 = m^2 + \alpha(1-\alpha) q^2$

$$= \frac{1}{(2\pi)^d} \frac{\pi^{d/2}}{(M^2)^{2-d/2}} \frac{\Gamma(2-d/2)}{\Gamma(2)}$$

$$= \frac{\pi^{d/2}}{(2\pi)^d} \frac{\Gamma(2-d/2)}{\Gamma(2)} \int_0^1 (m^2 + \alpha(1-\alpha)q^2)^{-(2-d/2)} d\alpha$$

or $$\int_0^1 \frac{\mu^{-2\epsilon}}{(4\pi)^2} \left(\frac{1}{\epsilon} - \gamma_E + \ln \frac{4\pi\mu^2}{m^2 + \alpha(1-\alpha) q^2} + O(\epsilon) \right) dx$$

$$= \frac{\mu^{-2\epsilon}}{(4\pi)^2} \left(\frac{1}{\epsilon} - \gamma_E - \int_0^1 d\alpha \ \ln \frac{(m^2 + (\alpha - \alpha^2) q^2)}{4\pi\mu^2} \right)$$

r_1 & r_2 roots of $(-\alpha^2 + \alpha) q^2 + m^2 \rightarrow -(\alpha - r_1)\cdot(\alpha - r_2)$

$$\frac{\mu^{-2\epsilon}}{(4\pi)^2} \left(\frac{1}{\epsilon} - \gamma_E + \underbrace{\int_0^1 d\alpha \left[\ln \frac{4\pi\mu^2}{-(\alpha - r_1)} + \ln \frac{4\pi\mu^2}{(\alpha - r_2)} \right]} \right)$$

$$\int_0^1 d\alpha \ \ln \frac{A}{\alpha - r_1} = \left[-A \left[\eta \ln \frac{(\alpha - r_1)}{A} - \left(\frac{\alpha - r_1}{A} \right) \right] \right]_0^1 , \quad \eta = \frac{(\alpha - r_1)}{A}$$

$$\frac{\mu^{-2\epsilon}}{(4\pi)^2} \left(\frac{1}{\epsilon} - \gamma_E \left[-(\alpha - r_1) \ln \frac{\alpha - r_1}{4\pi\mu^2} + \alpha - r_1 - (r_2 - \alpha) \ln \frac{r_2 - \alpha}{4\pi\mu^2} + r_2 - \alpha \right]_0^1 \right)$$

etc... ad lib

Problem set 6

3)
$$\int \frac{d^d p}{(2\pi)^d} \frac{1}{(p^2+m^2)^n} \quad n \text{ integer.}$$

$$= \frac{\pi^{d/2}}{(m^2)^{n-d/2}} \frac{\Gamma(n-d/2)}{\Gamma(n)}$$

$d = 4-2\epsilon$, n Integer

$$\frac{\pi^2}{(m^2)^{n-2+\epsilon}} \frac{\Gamma(n-2+\epsilon)}{\Gamma(n)} \leftarrow (n-1)!$$

$n=1$ $$\frac{\pi^2}{(m^2)^{-1+\epsilon}} \frac{\Gamma(-1+\epsilon)}{1} \quad -\frac{1}{\epsilon}\left(1+(1-\gamma_E)\epsilon+O(\epsilon^2)\right)$$

$$= \frac{\pi^2}{(m^2)^{-1}}\left(-\frac{1}{\epsilon}+\gamma_E-1+O(\epsilon)\right)$$

<u>Divergent</u> $\epsilon \to 0$

$n=2$ $$\frac{\pi^2}{(m^2)^\epsilon} \frac{\Gamma(\epsilon)}{2} = \frac{\pi^2}{2}\left(\frac{1}{\epsilon}-\gamma_E+O(\epsilon)\right)(m^2)^{-\epsilon}$$

$1-\epsilon \ln a + O(\epsilon^2)$

$$\frac{\pi^2}{2}\left(\frac{1}{\epsilon}-\ln a-\gamma_E+O(\epsilon)\right)$$

<u>Divergent</u> $\epsilon \to 0$

$n=3$ $$\frac{\pi^2}{(m^2)^{1+\epsilon}} \frac{\Gamma(1+\epsilon)}{6} = \frac{\pi^2}{m^2 6}\left(1+\gamma_E\epsilon+O(\epsilon^2)\right)$$

<u>Finite</u> $\epsilon \to 0$

$n>3$ Finite $\epsilon \to 0$

$n=0$? $$\frac{\pi^2}{(m^2)^{-2+\epsilon}} \frac{\Gamma(-2+\epsilon)}{} = \frac{\pi^2}{(m^2)^{-2}} \frac{-1}{\epsilon-2}\Gamma(\epsilon-1) = \ldots$$

<u>Divergent</u> $\epsilon \to 0$

Problem set 7

1) Solve

$$\mu \frac{d}{d\mu} \lambda_R = \frac{3}{(4\pi)^2} \lambda_R^2$$

$$\frac{3}{(4\pi)^2} \frac{d\mu}{\mu} = \frac{d\lambda_R}{\lambda_R^2} \Rightarrow -\frac{1}{\lambda_R} = \frac{3}{(4\pi)^2} \ln \mu + c$$

$$\lambda_R = -\frac{(4\pi)^2}{3 \ln \mu/\mu_0}$$

Problem set 7

2. Φ^4 theory: show $Z_\phi = 1 + \mathcal{O}(\lambda^2)$

$$G^{(n)}_{B,\bar{c}} = Z_\phi^{-n/2}\, G^{(n)}_{R,\bar{c}}$$

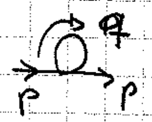
2-point function $\quad G^{2}_{B,\bar{c}} = Z_\phi^{-1} G^{2}_{R,\bar{c}}$

Error!!

$$\Rightarrow \quad (p^2 + m_B^2 + \Pi_B) = Z_\phi^{-1}(p^2 + m_R^2 + \Pi_R)$$

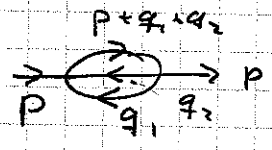
if Π_B & Π_R independent of p

then matching $p^2 = Z_\phi^{-1} p^2 \Rightarrow Z_\phi^{-1} = 1 = Z_\phi$

$\Pi_B = c + p^2\,\mathcal{O}(\lambda^2)$ because at order λ,
only diagram is 

which is independent of p.

$$\Rightarrow \quad Z_\phi^{-1} \approx 1 + \mathcal{O}(\lambda^2) \Rightarrow Z_\phi = 1 + \mathcal{O}(\lambda^2)$$

2-loop 

depends on p!!

error in problem sheet!!

3) Show $u^\dagger(p,s)\, u(p,s') = 2E_p \delta_{ss'}$

$$u(p,s) = \begin{pmatrix} \sqrt{p\cdot\sigma}\; \xi_s \\ \sqrt{p\cdot\bar{\sigma}}\; \xi_s \end{pmatrix} \qquad \begin{aligned} \sigma &= (1, \vec{\sigma}) \\ \bar{\sigma} &= (1, -\vec{\sigma}) \end{aligned}$$

$$u^\dagger u = \left(\sqrt{p\cdot\sigma}\,\xi_s,\ \sqrt{p\cdot\bar{\sigma}}\,\xi_s\right)\begin{pmatrix} \sqrt{p\cdot\sigma}\; \xi_{s'} \\ \sqrt{p\cdot\bar{\sigma}}\; \xi_{s'} \end{pmatrix}$$

Normalization $\xi_s \xi_{s'} = \delta_{ss'}$

$$= p\cdot\sigma\, \xi_s \xi_{s'} + p\cdot\bar{\sigma}\, \xi_s \xi_{s'}$$

$$= 2p_0\, \delta_{ss'} = \underline{\underline{2E_p \delta_{ss'}}}$$

Problem set 8

1) $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$

~~a) $\mathrm{Tr}\{\gamma^5\} = 0$~~

$\{\gamma^5, \gamma^\mu\} = 0$

c) $\mathrm{Tr}[\gamma^5\gamma^\mu] = -\mathrm{Tr}\{\gamma^\mu\gamma^5\}$
$= \mathrm{Tr}\{\gamma^\mu\gamma^5\}$ $\Big\} = 0$

b) $\mathrm{Tr}\{\underbrace{\gamma\gamma\cdots\gamma}_{n \text{ odd}}\}$

(Otherwise, use 1,2,3 $\neq \mu$)

$\mathrm{Tr}\{\gamma^\mu\} = \mathrm{Tr}[\gamma^\mu\gamma^0\gamma^0]$ $(\mu \neq 0)$

$= -\mathrm{Tr}\{\gamma^0\gamma^\mu\gamma^0\} = \mathrm{Tr}[\gamma^\mu\gamma^0\gamma^0] = 0$

~~$\mathrm{Tr}\{\gamma \cdots \gamma^\mu\} = \mathrm{Tr}[\gamma \cdots \gamma^\mu]$ past~~

$\mathrm{Tr}[\gamma^a\gamma^b\gamma^c]$ choose $\mu \neq a, b, c$

$\|$

$\mathrm{Tr}\{\gamma^a\gamma^b\gamma^c\gamma^\mu\gamma^\mu\}$

$\|$ $\Rightarrow 0$

$-\mathrm{Tr}[\gamma^\mu\gamma^a\gamma^b\gamma^c\gamma^\mu] = -\mathrm{Tr}\{\gamma^a\gamma^b\gamma^c\gamma^\mu\gamma^\mu\}$

if $n > 3$ & odd since there are only 4 different γ-matrices, 2 must be the same & reduces to $n = 3$

a) $\mathrm{Tr}\{\gamma^5\} = \mathrm{Tr}\{\gamma^5\gamma^0\gamma^0\} = -\mathrm{Tr}\{\gamma^5\gamma^0 = 0$

d) $\mathrm{Tr}\{\gamma^5\gamma^\mu\gamma^\nu\} = \mathrm{Tr}\{\gamma^5\gamma^\mu\gamma^\nu\gamma^0\gamma^0\}$ $\mu, \nu \neq 0$ otherwise choose $0 \to 1$

$= -\mathrm{Tr}\{\gamma^0\gamma^5\gamma^\mu\gamma^\nu\gamma^0\} = -\mathrm{Tr}\{\gamma^5\gamma^\mu\gamma^\nu\}$

$= 0$

Problem set 8

2) $c = (c_1\ c_2)^T \quad c^+ = (c_1^\wedge\ c_2)$

grassmann, M 2x2 complex matrix

$$\int dc_1^\wedge dc_1\, dc_2^\wedge dc_2\, \underbrace{\exp(-c^+Mc)}_{1 - c^+Mc + (c^+Mc)^2 \ldots}$$

$$1 \begin{pmatrix} - c_1^\wedge m_{11} c_1 - c_2^\wedge m_{22} c_2 \\ - c_1^\wedge m_{12} c_2 - c_2^\wedge m_{21} c_1 \end{pmatrix}$$

$$+ \frac{(\quad)^2}{2} \cancel{(\ldots)}$$

Only terms that survive have all $c_1, c_1^\wedge, c_2, c_2^\wedge$

So must be (note $\frac{1}{2}$ cancels factor 2)

$$\int dc_1^\wedge dc_1\, dc_2^\wedge dc_2 \begin{pmatrix} c_1^\wedge m_{11} c_1\, c_2^\wedge m_{22} c_2 \\ + c_1^\wedge m_{12} c_2\, c_2^\wedge m_{21} c_1 \end{pmatrix}$$

Remember signs $(+1)^6\, m_{11} m_{22}\ (-1)^3\, m_{12} m_{21}$

$$= m_{11} m_{22} - m_{12} m_{21} = \underline{\underline{\det M}}$$

3) as above

$$\langle \ldots \rangle = \frac{\int (\ldots)\, e^{-c^+Mc}}{\int e^{-c^+Mc}} \leftarrow \det M$$

a) $\langle c_1 c_\ell^\wedge \rangle \Rightarrow \int dc_1^\wedge dc_1 dc_2^\wedge dc_2 \begin{pmatrix} 1 - c_1^\wedge m_{11} c_1 \\ - c_1^\wedge m_{12} c_2 \\ - c_2^\wedge m_{21} c_1 \\ - c_2^\wedge m_{22} c_2 \end{pmatrix} \cdot c_1 c_\ell^\wedge$

$\ell = 1, 2$

Only term $\int \ldots\ \cancel{c_2}\ - c_1 c_\ell^\wedge c_{\bar\ell}^\wedge m_{\bar\ell 2} c_2$

$\ell = 1\ + m_{22}$; $\ell = 2\ - m_{12}$ $\quad \underline{/ \det M}$

b) $\langle c_1 c_1 \rangle = 0$, Trivial

c) $\langle c_1 c_2 c_1^\wedge c_2^\wedge \rangle = \int dc_1^\wedge dc_1\, dc_2^\wedge dc_2\ c_1 c_2 c_1^\wedge c_2^\wedge$

$= -1$

Problem set 9

1) f^{abc} structure constants of $SU(N_c)$

$$[T^a, T^b] = i f^{abc} T^c \qquad Tr[T^a T^b] = \frac{1}{2}\delta^{ab}$$

antisymmetric in abc permutation

$$Tr\{T^d [T^a, T^b]\} = i f^{abc} Tr[T^c T^d]$$
$$= i \frac{1}{2} f^{abd}$$

$$= Tr[T^d T^a T^b - T^d T^b T^a]$$

$a \leftrightarrow b$ obvious

$\underline{\underline{a \leftrightarrow d}}$

$$\frac{i}{2} f^{dab} \quad Tr[T^a T^d T^b - T^a T^b T^d]$$
$$\Rightarrow (\text{cyclic}) \quad Tr[T^d T^b T^a - T^d T^a T^b] = -f^{adb} \frac{i}{2}$$

2) $\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F^{\mu\nu a} \qquad F = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c$

classical equations of motion

$$\frac{\delta}{\delta A_\alpha^d} = -F^{\alpha\nu a} g f^{adb} A_\nu^b$$
$$\frac{\delta}{\delta \partial_\beta A_\alpha^d} = -F^{\beta\alpha d}$$

E-L $\Rightarrow$

$$\left[\frac{\delta}{\delta A_\alpha^d} - \partial_\beta \frac{\delta}{\delta \partial_\beta A_\alpha^d}\right]\mathcal{L} = 0$$

$$= -F^{\alpha\nu a} g f^{adb} A_\nu^b + \partial_\beta F^{\beta\alpha d}$$

$$\boxed{\left[\delta^{ad}\partial_\beta - g f^{abd} A_\beta^b\right] F^{\beta\alpha d} = 0}$$

$$D_\beta^{ad} F^{\beta\alpha d} = 0$$

Problem set 9

3) Wilson Line defined by

$$D_\mu(x_1)\, W(x_1, x_2) = 0$$
$$W(x_1, x_2) = 0$$

a) formal solution

$$(\partial_\mu - i A_\mu(x_1))\, W(x_1, x_2) = 0$$

$$\Rightarrow W(x_1, x_2) = e^{i\int_{x_2}^{x_1} A_\mu(x)\, dx^\mu}$$

: Boundary conditions

$$\Rightarrow W(x_1, x_2)\big|_{x_1 = x_2} = e^0 = 1$$

b) Under gauge transformations

$$D'_\mu(x_1) = U(x_1)\, D_\mu(x_1)\, U^\dagger(x_1)$$

Want $D'_\mu(x_1)\, W'(x_1, x_2) = 0$

& $W'(x_1, x_2) = 1, \; x_1 = x_2$

$$\Rightarrow U_{x_1} D_\mu(x_1)\, U^\dagger(x_1)\, W'(x_1, x_2) = 0$$

$$\Rightarrow W'(x_1, x_2) = U(x_1)\, W'(x_1, x_2)\, U^\dagger(x_2)$$

Boundary condition

Since at $x_1 = x_2 \Rightarrow U(x_1)\, W(x_1, x_1)\, U^\dagger(x_1) = 1$

1 also