QUANTUM FIELD THEORY

- Why (should we care)?
  - According to current knowledge, QFTs describe the fundamental forces and particles

    Standard Model $\left\{ \begin{array}{l} \text{QED, Quantum Electrodynamics} \\ \text{QCD, strong interactions} \\ \text{Electroweak} \end{array} \right.$

- Quantum mechanics $\longleftrightarrow$ QFT ?
  - QM: 1 (or more) particles, degrees of freedom: coordinates $\bar{x}_i$

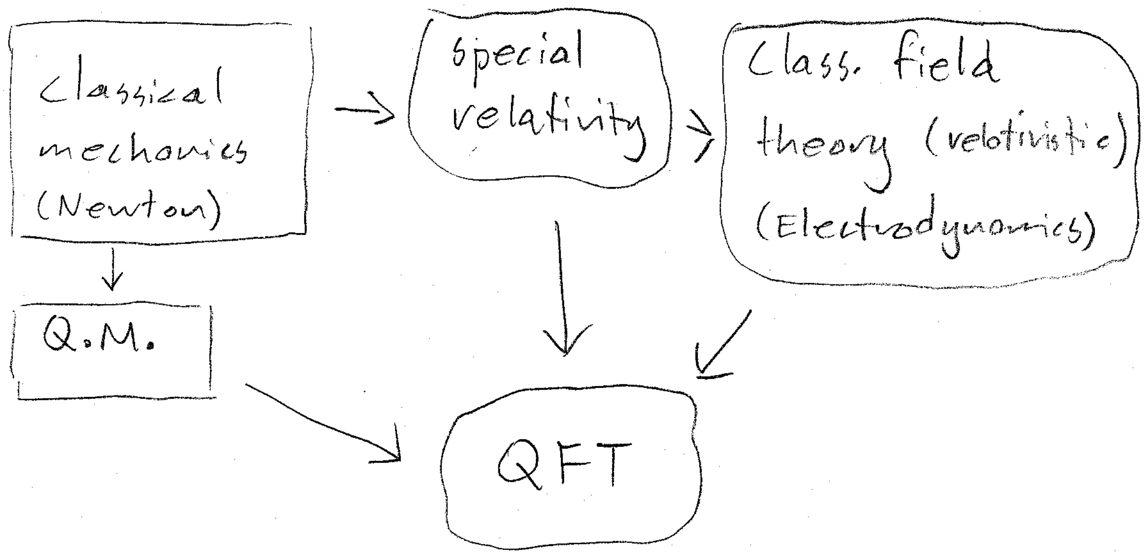
    $$H = \sum_i \frac{\bar{p}_i^2}{2m} + V(\bar{x}_i)$$

  - QFT: Degrees of freedom: field $\varphi(\bar{x})$: at every point a new d.o.f

    "particle:" moving (quantized) wave of the field

- Automatically a many-particle theory - many waves
- QFT's correctly combine
  * Quantum mechanics
  * Special theory of relativity

  } QFT
- Relativistic quantum theories are many-particle theories: for each particle type, ∃ <u>anti</u>p<u>article</u>
  - particle - antiparticle pair creation



- General theory of relativity; e.g. Gravity ? Difficult to include in QFTs in 4 dimensions

Books:

* * Peskin, Schroder: An Introduction to Quantum Field Theory
* \- Weinberg: The Quantum Theory of Fields I+II
* \- Bailin, Love: Introduction to Gauge Field Theory
* * Zee: QFT in a nutshell
* \- Itzykson, Zuber: QFT

Units:

$$\hbar = c = 1$$

$[c] = \frac{m}{s}$

$[\hbar] = kg \cdot m/s$

$$[\text{length}] = [\text{time}] = [\text{energy}]^{-1} = [\text{mass}]^{-1}$$

• Answers usually in terms of energy (GeV) or (fm) $= 10^{-15}$ m

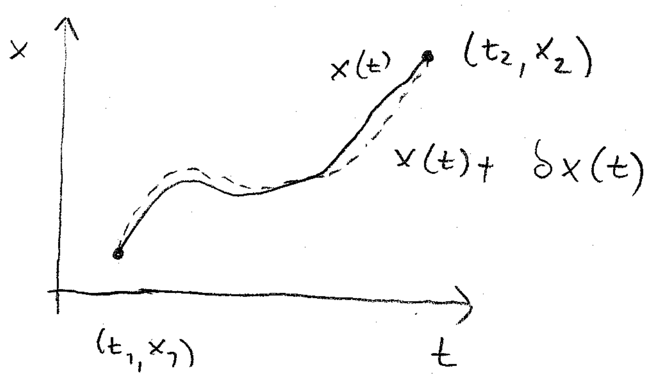
0.2 GeV $= 1/\text{fm}$ or $0.2\,\text{GeV} \cdot \frac{1}{\hbar c} = \frac{1}{\text{fm}}$

1. Canonical quantization of scalar fields

1.1 Classical field theory: Lagrange and Hamilton-formalism

- Recall: classical particle (non-relativistic)
  - Kinetic energy $T = \frac{1}{2} m \dot{x}^2$
  - Potential energy $V = V(x)$
  - Lagrange function (Lagrangian)

    $$L = T - V = \frac{1}{2} m \dot{x}^2 - V(x) = L(x, \dot{x})$$
  - Action $S = \int_{t_1}^{t_2} dt \, L(x, \dot{x})$ (1.1)
  - Principle of extremal action: particle moves along trajectory, which is a minimum (maximum) of S



Variation: $x(t) \to x'(t) = x(t) + \delta x(t)$

boundary condition $\delta x(t_1) = \delta x(t_2) = 0$

$$S \to S' = \int_{t_1}^{t_2} dt\, L(x+\delta x,\ \dot{x}+\delta\dot{x})$$

$$= \int_{t_1}^{t_2} dt \left(\frac{\partial L}{\partial x} \cdot \delta x + \frac{\partial L}{\partial \dot{x}} \overbrace{\delta\dot{x}}^{\frac{d}{dt}\delta x} \right) + S$$

$$= S + \underbrace{\int_{t_1}^{t_2} dt \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}} \delta x \right)}_{0 \text{ because } \delta x(t_1) = \delta x(t_2) = 0} + \int_{t_1}^{t_2} dt \left(\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} \right) \delta x \qquad (1.2)$$

In extremum $S' = S + \delta S = S$, thus
the last integral must vanish for all $\delta x(t)$

$$\Rightarrow \boxed{\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0}$$

Euler-Lagrange equation (1.3)

Using $L = \frac{1}{2} m \dot{x}^2 - V(x) \Rightarrow m\ddot{x} = -\frac{\partial V}{\partial x}$

$ma = F$

other boundary conditions than $x(t_1), x(t_2)$ fixed are also possible, for example $x(t_1), \dot{x}(t_1)$ fixed. Lead to same E-L eqn.

Hamiltonian formulation:

- Canonical momentum $p \equiv \dfrac{\partial L}{\partial \dot{x}} = m\dot{x}$ (1.4)
- Hamiltonian

$$H(x,p) \equiv p\dot{x} - L \qquad \text{Legendre transformation}$$
$$= \frac{p^2}{2m} + V(x)$$

- Hamilton EQM:

$$\boxed{\begin{aligned} \dot{x} &= \frac{\partial H}{\partial p} \\ \dot{p} &= -\frac{\partial H}{\partial x} \end{aligned}} \Rightarrow \begin{aligned} \dot{x} &= p/m \\ \dot{p} &= -\frac{\partial V}{\partial x} \end{aligned} \quad \text{OK!} \qquad (1.5)$$

Both Hamilton and Lagrange-formalisms are equivalent w. Newton mechanics. However, they offer better starting point for quantization:

Hamilton → canonical quantization

Lagrange → path integral quantization

Generalize to classical fields

$x \to \varphi(\bar{x})$ ∞ number of degrees of freedom

$t \to t = x^0 \; ; \; \varphi(x^0, x^i)$ coordinates

($x_0 = ct$, but choose $c = 1$)

Metric tensor $g^{\mu\nu} = \begin{pmatrix} +1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix} = g_{\mu\nu}$

$$x_\mu = g_{\mu\nu} x^\nu \equiv \sum_\nu g_{\mu\nu} x^\nu$$

(Einstein convention)

$$\partial_\mu = \frac{\partial}{\partial x^\mu} \; ; \; \partial^\mu = \frac{\partial}{\partial x_\mu}$$

Thus, instead of single coordinate x, we have a "field" of variables $\varphi(x^0, \bar{x})$ defined at all spatial points $\bar{x}$, and whose time (x^0) dependence we want to study

Now

- $\mathcal{L}(\varphi, \partial_\mu \varphi)$: Lagrange density
- $L[\varphi, \partial_\mu \varphi] \equiv \int_V d^3\bar{x}\, \mathcal{L}(\varphi, \partial_\mu \varphi)$ Lagrange function
- $S = \int_{t_1}^{t_2} dt\, L[\varphi, \partial_\mu \varphi]$ Action

$$= \int_\Omega d^4x\, \mathcal{L}(\varphi, \partial_\mu \varphi) \qquad (1.6)$$

Ω " 4-volume $= [t_1, t_2] \times V$

Extremizing :

$$S' = \int_\Omega d^4x \, \mathcal{L}(\varphi', \partial_\mu \varphi') = S + \delta S$$

$$\simeq S + \int_\Omega d^4x \left[\frac{\partial \mathcal{L}}{\partial \varphi} \delta\varphi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \delta(\partial_\mu \varphi) \right]$$

$$\delta(\partial_\mu \varphi) = \partial_\mu \varphi' - \partial_\mu \varphi = \partial_\mu \delta\varphi$$

$$= S + \int_\Omega d^4x \left[\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right) \right] \delta\varphi$$

$$+ \int_\Omega d^4x \, \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \delta\varphi \right]$$

The last term = $\int_{\partial\Omega} dS \, n_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \delta\varphi \right] = 0$ ← boundary of Ω

because $\delta\varphi = 0$ on boundary.

$$\delta S = 0 \quad \Rightarrow \quad \boxed{\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right] = 0} \qquad (1.7)$$

Euler-Lagrange equation

Example: $\mathcal{L} = \frac{1}{2} \partial^\mu \varphi \, \partial_\mu \varphi - \frac{1}{2} m^2 \varphi^2$ | Scalar field

$$\Rightarrow \left[\partial_\mu \partial^\mu + m^2 \right] \varphi = 0 \qquad (1.8)$$

Klein-Gordon equation or relativistic wave equation

- This generalizes easily to <u>N components</u>

$$\mathcal{L} = \frac{1}{2}\partial^\mu \varphi^a \partial_\mu \varphi^a - V(\varphi)$$

(implicit μ, a - sums, $a = 1..N$)

$$\Rightarrow \quad \frac{\partial \mathcal{L}}{\partial \varphi^a} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi^a)} = 0 \qquad (1.9)$$

- If $V(\varphi) = V(\varphi^a \varphi^a)$, action is $O(N)$-invariant:

$\varphi \to \varphi' = M\varphi$; $M^T M = 1$; M is $N\times N$ orthogonal matrix

- $N=2$ gives <u>complex</u> field

$$\phi = \frac{1}{\sqrt{2}}(\varphi^1 + i\varphi^2)$$

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - V(\phi^*\phi) \qquad (1.10)$$

complex field lagrangian customarily without $\frac{1}{2}$-factor

Eqn of motion by formally (Homework!)

$$\frac{\partial \mathcal{L}}{\partial \phi^*} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} = 0 \quad , \quad \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = 0$$

$$\Rightarrow \partial_\mu \partial^\mu \phi + m^2 \phi \qquad \text{if } V = m^2 \phi^* \phi$$

$$\partial_\mu \partial^\mu \phi^* + m^2 \phi^*$$

In <u>Hamiltonian</u> formalism we define

- Canonical momentum $\pi(x) \equiv \frac{\partial \mathcal{L}}{\partial \dot{\varphi}(x)} = \frac{\partial \mathcal{L}}{\partial(\partial_0 \varphi)}$

$$\dot{\varphi} = \partial_0 \varphi = \partial^0 \varphi$$

- Hamiltonian density

$$\mathcal{H}(\pi, \varphi) = \pi \partial_0 \varphi - \mathcal{L}$$
$$= \frac{1}{2}\pi^2 + \frac{1}{2}\sum_{i=1}^{3}(\partial_i \varphi)^2 + \frac{1}{2} m \varphi^2$$

- EQNs: $\dot{\pi}(x) = -\frac{\partial H}{\partial \varphi(x)}, \quad \dot{\varphi}(x) = \frac{\partial H}{\partial \pi(x)}$

Note that in Lagrange-formalism time (x^0) has no special status

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<u>Noether's theorem</u> [Emmy Noether 1918]

- If $\exists$ transformations $\varphi \to \varphi + \delta\varphi$ and/or $x \to x + \delta x$ which leave S invariant (for any subvolume) there exists a conserved <u>Noether current</u> j^μ ; $\boxed{\partial_\mu j^\mu = 0}$ (1.11)

$\rho = j^0(x)$: "charge" density

$j^i(x)$: current density

* Invariance in coordinate transformations (translations, rotations) – property of almost any Lagrange density

$\Rightarrow \quad \partial_\mu T^{\mu\nu} = 0$, $T^{\mu\nu}$ = energy-momentum tensor

$T^{00} = \mathcal{H}$, $T^{i0} = p^i$ = momentum density

$\partial_\mu T^{\mu 0} = 0$: 4-momentum conservation

$\partial_\mu T^{\mu i} = 0$: Angular momentum conservation

* Internal symmetry; example $\varphi(x) \in \mathbb{C}$:

$$\mathcal{L} = (\partial_\mu \varphi^*)(\partial^\mu \varphi) - m^2 \varphi^* \varphi$$

invariant $\varphi \to e^{i\theta}\varphi$; $\varphi^* \to e^{-i\theta}\varphi^*$, θ const.

Leads to conserved current, "charge"

$$j_\mu = \mathrm{Im}\, \varphi^* \partial_\mu \varphi = -\frac{i}{2}(\varphi^* \partial_\mu \varphi - \varphi \partial_\mu \varphi^*) \qquad (1.12)$$

$$\partial_\mu j^\mu = 0$$

Derivation of Noether's theorem
(for internal symmetry)

Let $\varphi^a(x) \to \varphi^{a'}(x) = \varphi^a(x) + \lambda^a_i(x)\, \underbrace{\delta\omega^i}_{\text{infinitesimal}}$; $a = 1..N$, $i = 1..M \le N$

be a transformation which leaves

S invariant :

$$\delta S = 0 = \int d^4x\, \delta\mathcal{L} \Rightarrow \delta\mathcal{L} = \partial_\mu b^\mu, \quad \text{some vector}$$

so that EQM invariant.

Assume here $b^\mu = 0$.

$$\delta\mathcal{L} = 0 = \underbrace{\frac{\partial\mathcal{L}}{\partial\varphi^a}}_{= \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi^a)}} \lambda^a_i\, \delta\omega^i + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi^a)} \underbrace{\delta(\partial_\mu\varphi^a)}_{= \partial_\mu(\varphi^{a'} - \varphi^a) = \partial_\mu\lambda^a_i\,\delta\omega^i}$$

$$= \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi^a)}\lambda^a_i\right]\delta\omega^i = 0$$

$$\Rightarrow \boxed{j^\mu_i = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi^a)}\lambda^a_i} \quad \text{is conserved current} \qquad (1.13)$$

(note: i not a vector index)

$$\partial_\mu j^\mu = 0 = \partial_0 j^0 + \partial_i j^i = \partial_t j^0 - \vec{\nabla}\cdot\vec{j}$$

$$Q = \int_V d^3\bar{x}\, j^0$$

Invariance in coordinate transformations

$$x^\mu \to x^{\mu'} = x^\mu + \delta x^\mu$$

(translations and rotations)

Now $\varphi(x) \to \varphi(x') = \varphi + (\partial_\mu \varphi)\cdot \delta x^\mu$, and $\delta\mathcal{L}$ is $\neq 0$:

$$\delta\mathcal{L} = \mathcal{L}(x') - \mathcal{L}(x) = \delta x^\mu \partial_\mu \mathcal{L}$$

$$= \frac{\partial \mathcal{L}}{\partial \varphi}\,\delta\varphi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)}\,\delta(\partial_\mu \varphi)$$

$$\frac{\partial \mathcal{L}}{\partial \varphi} \to \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)}$$

$$\delta\varphi \to \partial_\nu \varphi\, \delta x^\nu$$

$$\delta(\partial_\mu\varphi) \to \partial_\mu' \varphi' - \partial_\mu \varphi$$

$$= \frac{\partial x^\nu}{\partial x^{\mu'}}\,\partial_\nu \varphi' - \partial_\mu \varphi$$

$$= \left(\delta^\nu_\mu - \partial_\mu(\delta x^\nu)\right)\partial_\nu \varphi' - \partial_\mu \varphi + O(\delta^2)$$

$$= \partial_\mu \delta\varphi - \partial_\mu(\delta x^\nu)\,\partial_\nu \varphi + O(\delta^2)$$

$$= \partial_\mu(\partial_\nu \varphi \cdot \delta x^\nu) - \partial_\mu(\delta x^\nu)\,\partial_\nu\varphi + O(\delta^2)$$

$$= (\partial_\mu \partial_\nu \varphi)\cdot \delta x^\nu + O(\delta^2)$$

$$\Rightarrow \delta x_\nu\, \partial_\mu \left[-g^{\mu\nu}\mathcal{L} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)}\,\partial^\nu \varphi \right] = 0 \qquad (1.14)$$

$$\Rightarrow T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)}\,\partial^\nu\varphi - g^{\mu\nu}\mathcal{L}$$

Energy-momentum tensor (1.15)

$$\partial_\mu T^{\mu\nu} = 0$$

4 cons. currents, $\nu = 0,1,2,3$

1.2. Fundamentals of quantization

- Consider again classical particle in Hamilton-formalism

$$\dot{x} = \frac{\partial H}{\partial p} \quad , \quad \dot{p} = -\frac{\partial H}{\partial x} \quad ; \quad H(x,p)$$

- Recall: Poisson brackets of quantities A, B:

$$[A,B]_p = \frac{\partial A}{\partial x}\frac{\partial B}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial B}{\partial x}$$

Here x, p on the same time, for example t=0.
Now holds

$$[x,x]_p = [p,p]_p = 0$$

$$[x,p]_p = 1$$

$$\dot{x}(t) = [x(t), H]_p$$

$$\dot{p}(t) = [p(t), H]_p$$

$$\dot{H} = [H,H]_p = 0 \qquad \text{energy conservation}$$

Quantize:

$$x \to \hat{x} \, , \quad p \to \hat{p} \qquad \text{operators}$$

$$\left.\begin{array}{l} [\hat{x},\hat{x}] = [\hat{p},\hat{p}] = 0 \\ [\hat{x},\hat{p}] = i \quad (= i\hbar) \end{array}\right\} \text{Equal time commutators}$$

$$i\frac{d}{dt}\hat{O}(t) = [\hat{O},\hat{H}] \qquad \text{Heisenberg-picture dynamics}$$

Reminder: Schrödinger- and Heisenberg-pictures

- In Schrödinger picture operators do not depend on time, states do: Schrödinger equation

$$i \frac{d}{dt} |t;s\rangle = \hat{H} |t;s\rangle \qquad (1.16)$$

$$\Rightarrow \quad |t;s\rangle = e^{-i\hat{H}t} |0;s\rangle$$

Expectation values $\langle t;s| \hat{O}_s |t;s\rangle = \langle 0;s| e^{i\hat{H}t} \hat{O}_s e^{-i\hat{H}t} |0;s\rangle$

- In Heisenberg picture operators depend on time, states do not:

$$|H\rangle \equiv |0;s\rangle$$

$$\hat{O}_H(t) \equiv e^{i\hat{H}t} \hat{O}_s e^{-i\hat{H}t}$$

Expectation values are as in Schrödinger:

$$\langle H| \hat{O}_H(t) |H\rangle = \langle t;s| \hat{O}_s |t;s\rangle$$

Heisenberg eqn. of motion

$$i \frac{d}{dt} \hat{O}_H(t) = [\hat{O}_H(t), \hat{H}]$$

- Physics is equivalent!

Fundamental example (MUST KNOW!):

Harmonic oscillator

$$\hat{H} = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{x}^2 \qquad m, \omega \text{ free parameters}$$

Standard notation:

$$\begin{cases} \hat{a} \equiv \sqrt{\frac{m\omega}{2}}\,\hat{x} + \frac{i}{\sqrt{2m\omega}}\,\hat{p} \\ \hat{a}^+ \equiv \sqrt{\frac{m\omega}{2}}\,\hat{x} - \frac{i}{\sqrt{2m\omega}}\,\hat{p} \end{cases} \iff \begin{aligned} \hat{x} &= \frac{1}{\sqrt{2m\omega}}(\hat{a}+\hat{a}^+) \\ \hat{p} &= i\sqrt{\frac{m\omega}{2}}(-\hat{a}+\hat{a}^+) \end{aligned}$$

$$[\hat{a}, \hat{a}^+] = -[\hat{a}^+, \hat{a}] = 1 \qquad \text{(Homework!)}$$

Classically:

$$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m} \quad ; \quad \dot{p} = -\frac{\partial H}{\partial x} = -m\omega^2 x$$

$$\Rightarrow \ddot{x} = -\omega^2 x \quad \Rightarrow \quad x(t) = \underbrace{x(0)}_{\text{initial}}\cos\omega t + \underbrace{\frac{p(0)}{m\omega}}_{\text{conditions}}\sin\omega t$$

Average $x^2(t)$:

$$\langle x(t)^2 \rangle = \frac{1}{2\pi/\omega}\int_0^{2\pi/\omega} dt\, x(t)^2 = \frac{1}{2}x(0)^2 + \frac{1}{2}\left[\frac{p(0)}{m\omega}\right]^2$$

In QM:

$$i\frac{d}{dt}\hat{x}_H = [\hat{x}_H, \hat{H}] = \frac{1}{2m}[\hat{x}_H, \hat{p}_H^2] \qquad \Big| \quad [A,BC] = [A,B]C + B[A,C]$$

$$= \frac{1}{2m}\left(\hat{p}[\hat{x},\hat{p}] + [\hat{x},\hat{p}]\hat{p}\right) = i\frac{\hat{p}_H}{m}$$

$$i\frac{d}{dt}\hat{p}_H = [\hat{p}_H, \hat{H}] = \frac{m\omega^2}{2}[\hat{p}_H, \hat{x}_H^2] = -m\omega^2\hat{x}_H$$

Thus, $\frac{d^2}{dt^2}\hat{x}_H = -\omega^2\hat{x}_H$ as in class, solution!

Solution: $\hat{x}_H(t) = \underset{\hat{x}_S}{\underset{\parallel}{\hat{x}(0)}}\cos\omega t + \frac{\hat{p}(0)}{m\omega}\sin\omega t$, $\hat{p}(0) = \hat{p}_S$

Likewise, $\hat{p}_H(t) = -m\omega\,\hat{x}(0)\sin\omega t + \hat{p}(0)\cos\omega t$

Alternatively, using $\hat{a}, \hat{a}^+$

$$\hat{x}_H(t) = \frac{1}{\sqrt{2m\omega}}\left(\hat{a}e^{-i\omega t} + \hat{a}^+e^{i\omega t}\right)$$

$$\hat{p}_H(t) = i\sqrt{\frac{m\omega}{2}}\left(-\hat{a}e^{-i\omega t} + \hat{a}^+e^{i\omega t}\right)$$

Expectation values:

$$\langle n|\hat{x}_H(t)^2|n\rangle = \frac{1}{2m\omega}\langle n|\left[\hat{a}\hat{a}e^{-2i\omega t} + \hat{a}\hat{a}^+ + \hat{a}^+\hat{a} + \hat{a}^+\hat{a}^+e^{2i\omega t}\right]|n\rangle$$

$$= \frac{1}{2m\omega}\left[\langle n|\hat{a}\sqrt{n+1}|n+1\rangle + \langle n|\hat{a}^+\sqrt{n}|n-1\rangle\right]$$

$$= \frac{2n+1}{2m\omega}$$

Here we used $\hat{a}^+|n\rangle = \sqrt{n+1}|n+1\rangle$

$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$

$|n\rangle$: eigenstate of $\hat{H}$

Note that non-equal time commutators of $\hat{x}_H(t)$ do not vanish:

$$[\hat{x}_H(t), \hat{x}_H(0)] = \frac{1}{2m\omega}\left([\hat{a}, \hat{a}^+]e^{-i\omega t} + [\hat{a}^+, \hat{a}]e^{i\omega t}\right)$$

$$= \frac{1}{2m\omega}\left(e^{-i\omega t} - e^{i\omega t}\right) = \frac{1}{im\omega} \sin \omega t$$

Note: also classical Poisson bracket

$$[x(t), x(0)]_P = [x(0) \cos \omega t, x(0)]_P$$

$$+ [\frac{p(0)}{m\omega} \sin \omega t, x(0)]_P = -\frac{1}{m\omega} \sin \omega t$$

$$\neq 0 \; !$$