1.3. Canonical quantization of fields

pg. 7:

$$H = \int d^3\bar{x} \left[\frac{1}{2}\pi^2 + \frac{1}{2}\partial_i\varphi\,\partial_i\varphi + \frac{1}{2}m\,\varphi^2 \right]$$

$$\pi(x) = \partial_0\varphi(x)$$

Generalize the steps from pg. 11:

- $\varphi(x) \longrightarrow \hat{\varphi}(x)$

  $\pi(x) \longrightarrow \hat{\pi}(x)$

  $H \longrightarrow \hat{H}$

- Postulate canonical commutator relations: <u>equal time</u>

  $$[\hat{\varphi}(x^0,\bar{x}), \hat{\varphi}(x^0,\bar{y})] = 0$$

  $$[\hat{\pi}(x^0,\bar{x}), \hat{\pi}(x^0,\bar{y})] = 0$$

  equivalent with $[\hat{x}_a, \hat{p}_b] = i\,\delta_{ab}$

  $$[\hat{\varphi}(x^0,\bar{x}), \hat{\pi}(x^0,\bar{y})] = i\,\delta^{(3)}(\bar{x}-\bar{y}) \qquad (1.17)$$

- Dynamics from Heisenberg:

  $$i\,\partial_0\hat{\varphi}(x^0,\bar{x}) \equiv [\hat{\varphi}(x^0,\bar{x}), \hat{H}]$$

  $$i\,\partial_0\hat{\pi}(x^0,\bar{x}) \equiv [\hat{\pi}(x^0,\bar{x}), \hat{H}]$$

  defines time evolution!

In analogy with standard Hamilton equations of motion, we obtain (homework!)

$$[\hat{\varphi}(x^0,\bar{x}),\ \hat{H}] = i\,\hat{\pi}(x^0,\bar{x}) \qquad (1.18)$$

$$[\hat{\pi}(x^0,\bar{x}),\ \hat{H}] = -i(-\vec{\nabla}^2+m^2)\,\hat{\varphi}(x^0,\bar{x})$$

In the latter one it is useful to note that $\hat{H}$ can be written (after partial integration)

$$\hat{H} = \int d^3\bar{x}\left[\frac{1}{2}\hat{\pi}^2+\frac{1}{2}\hat{\varphi}(-\bar{\nabla}^2+m^2)\varphi\right] \qquad (1.19)$$

Thus, we obtain

$$\partial_0^2\hat{\varphi}(x^0,\bar{x}) = \partial_0\hat{\pi}(x^0,\bar{x}) = (\nabla^2+m^2)\,\hat{\varphi}(x^0,\bar{x})$$

$$\Rightarrow [\partial_0^2-\bar{\nabla}^2+m^2]\,\hat{\varphi}(x^0,\bar{x}) = 0$$

Also $\hat{\varphi}(x^0,\bar{x})$ obeys Klein-Gordon equation, like the classical solution!

As usual with wave eqn., the solution can be written as a linear combination of plane waves : Ansatz

$$\hat{\varphi} = \hat{N}_p e^{-ip_\mu x^\mu} \quad \Rightarrow \quad (-p_0^2 + \vec{p}^{\,2} + m^2) N_p = 0$$

$$\Rightarrow \quad p_0 = \pm\sqrt{\bar{p}^2 + m^2}$$

Def. $E_{\bar{p}} \equiv \sqrt{\bar{p}^2 + m^2}$, $p_0 = \pm E_{\bar{p}}$

General solution :

$$\hat{\varphi} = \int d^3\bar{p} \left[\hat{N}^1_{(\bar{p})} e^{-iE_{\bar{p}}x^0 + i\bar{p}\cdot\bar{x}} + \hat{N}^2_{\bar{p}} e^{iE_{\bar{p}}x^0 + i\bar{p}\cdot\bar{x}} \right] \tag{1.20}$$

switch here $\bar{p} \to -\bar{p}$

$$E_p x^0 - \bar{p}\cdot\bar{x} = p_\mu x^\mu \equiv p\cdot x$$

$$p_0 \equiv E_p \geq 0$$

Convenient normalization:

$$\begin{cases} \hat{N}^1_{\bar{p}} = \dfrac{1}{\sqrt{(2\pi)^3 2E_{\bar{p}}}} \hat{a}_{\bar{p}} \\[2ex] \hat{N}^2_{\bar{p}} = \dfrac{1}{\sqrt{(2\pi)^3 2E_{\bar{p}}}} \hat{a}^+_{\bar{p}} \end{cases} \Rightarrow \tag{1.21}$$

$$\hat{\varphi}(x^0, \bar{x}) = \int \frac{d^3\bar{p}}{\sqrt{(2\pi)^3 2E_{\bar{p}}}} \left[\hat{a}_{\bar{p}} e^{-ip\cdot x} + \hat{a}^+_{\bar{p}} e^{ip\cdot x} \right] \tag{1.22}$$

Why the strange normalization (1.21)?

It makes the commutation relations of $\hat{a}_{\bar{p}}$, $\hat{a}^{\dagger}_{\bar{p}}$ particularly simple:

$$[\hat{a}_{\bar{p}}, \hat{a}_{\bar{q}}] = [\hat{a}^{\dagger}_{\bar{p}}, \hat{a}^{\dagger}_{\bar{q}}] = 0$$

$$[\hat{a}_{\bar{p}}, \hat{a}^{\dagger}_{\bar{q}}] = \delta^{(3)}(\bar{p}-\bar{q}) \qquad (1.23)$$

Proof: show that (1.23) $\Rightarrow$ (1.17)

$$(1.22) \Rightarrow \partial_0 \hat{\varphi} = \int_{\bar{q}} \left[-iE_{\bar{q}}\, \hat{a}_{\bar{q}}\, e^{-iq\cdot x} + iE_{\bar{q}}\, \hat{a}^{\dagger}_{\bar{q}}\, e^{+iq\cdot x} \right]$$

$$\int_{\bar{q}} \equiv \int \frac{d^3\bar{q}}{\sqrt{(2\pi)^3 2E_{\bar{q}}}}$$

$$\Rightarrow [\hat{\varphi}(x), \partial_0 \varphi(y)] = \int_{\bar{p}}\int_{\bar{q}} \Big\{ iE_{\bar{q}} \overbrace{[\hat{a}_{\bar{p}}, \hat{a}^{\dagger}_{\bar{q}}]}^{\delta^{(3)}(\bar{p}-\bar{q})} e^{-ip\cdot x + iq\cdot y} - iE_{\bar{q}} \underbrace{[\hat{a}^{\dagger}_{\bar{p}}, \hat{a}_{\bar{q}}]}_{-\delta^{(3)}(\bar{p}-\bar{q})} e^{ip\cdot x - iq\cdot y} \Big\}$$

$$= \int \frac{d^3\bar{p}}{(2\pi)^3 2E_{\bar{p}}} \Big\{ iE_{\bar{p}}\, e^{ip\cdot(y-x)} + iE_{\bar{p}}\, e^{ip\cdot(x-y)} \Big\}$$

Demand now $x^0 = y^0$ (page 16) $\Rightarrow$

$$= i \int \frac{d^3\bar{p}}{(2\pi)^3}\, e^{i\bar{p}\cdot(\bar{x}-\bar{y})} = i\,\delta^{(3)}(\bar{x}-\bar{y})$$

Ok!

Express $\hat{H} = \int d^3\bar{x} \left[\frac{1}{2}(\partial_0\hat{\varphi})(\partial_0\hat{\varphi}) + \frac{1}{2}(\partial_i\hat{\varphi})(\partial_i\hat{\varphi}) + \frac{1}{2}m^2\hat{\varphi}\hat{\varphi} \right]$

in terms of $\hat{a}_{\bar{p}}, \hat{a}^\dagger_{\bar{p}}$:

$$\underline{\hat{H} = \int d^3\bar{p}\, E_{\bar{p}} \frac{1}{2}\left(\hat{a}_{\bar{p}}\hat{a}^\dagger_{\bar{p}} + \hat{a}^\dagger_{\bar{p}}\hat{a}_{\bar{p}} \right)} \tag{1.24}$$

Proof:

$$\hat{H} = \int d^3\bar{x} \int_{\bar{p}}\int_{\bar{q}} \frac{1}{2}\Big\{ \left[-iE_{\bar{p}}\hat{a}_{\bar{p}}e^{-ip\cdot x} + iE_{\bar{p}}\hat{a}^\dagger_{\bar{p}}e^{ip\cdot x} \right]\left[\bar{p}\to\bar{q} \right] +$$
$$\left[ip_i\hat{a}_{\bar{p}}e^{-ip\cdot x} - ip_i\hat{a}^\dagger_{\bar{p}}e^{ip\cdot x} \right]\left[\bar{p}\to\bar{q} \right] +$$
$$m^2\left[\hat{a}_{\bar{p}}e^{-ip\cdot x} + \hat{a}^\dagger_{\bar{p}}e^{ip\cdot x} \right]\left[\bar{p}\to\bar{q} \right]\Big\}$$

$$= \frac{1}{2}\int d^3\bar{x}\int_{\bar{p}}\int_{\bar{q}} \Big\{ \left[-E_{\bar{p}}E_{\bar{q}} - \bar{p}\cdot\bar{q} + m^2 \right]\hat{a}_{\bar{p}}\hat{a}_{\bar{q}}\, e^{-i(p+q)\cdot x} +$$
$$\left[E_{\bar{p}}E_{\bar{q}} + \bar{p}\cdot\bar{q} + m^2 \right]\hat{a}_{\bar{p}}\hat{a}^\dagger_{\bar{q}}\, e^{-i(p-q)\cdot x} +$$
$$\left[E_{\bar{p}}E_{\bar{q}} + \bar{p}\cdot\bar{q} + m^2 \right]\hat{a}^\dagger_{\bar{p}}\hat{a}_{\bar{q}}\, e^{i(p-q)\cdot x} +$$
$$\left[-E_{\bar{p}}E_{\bar{q}} - \bar{p}\cdot\bar{q} + m^2 \right]\hat{a}^\dagger_{\bar{p}}\hat{a}^\dagger_{\bar{q}}\, e^{i(p+q)\cdot x} \Big\}$$

$$= \frac{1}{2}\int \frac{d^3\bar{p}}{2E_{\bar{p}}} \Big\{ \overset{=0}{\left[-E_{\bar{p}}^2 - \bar{p}^2 + m^2 \right]}\hat{a}_{\bar{p}}\hat{a}_{\bar{p}}\, e^{-i2E_{\bar{p}}x^0} +$$
$$\left[E_{\bar{p}}^2 + \bar{p}^2 + m^2 \right]\hat{a}_{\bar{p}}\hat{a}^\dagger_{\bar{p}} +$$
$$\left[E_{\bar{p}}^2 + \bar{p}^2 + m^2 \right]\hat{a}^\dagger_{\bar{p}}\hat{a}_{p} +$$
$$\underset{=0}{\left[-E_{\bar{p}}^2 - \bar{p}^2 + m^2 \right]}\hat{a}^\dagger_{\bar{p}}\hat{a}^\dagger_{\bar{p}}\, e^{i2E_{\bar{p}}x^0} \Big\}$$

$$= \frac{1}{2}\int d^3\bar{p}\, E_{\bar{p}} \left(\hat{a}_{\bar{p}} \hat{a}_{\bar{p}}^{+} + \hat{a}_{\bar{p}}^{+} \hat{a}_{\bar{p}} \right) \qquad \square$$

Operators $\hat{a}^{+}_{\bar{p}}$ and $\hat{a}_{\bar{p}}$ can be interpreted as <u>creation</u> and <u>annihilation</u> operators for a particle with momentum $\bar{p}$. They generate a <u>Fock-space</u> of n-particle-states:

We observe $\quad ([AB, C] = A[B,C] + [A,C]B)$

$$[\hat{H}, \hat{a}_{\bar{p}}] = \frac{1}{2}\int d^3\bar{p}'\, E_{\bar{p}'} \left([\hat{a}_{\bar{p}'} \hat{a}^{+}_{\bar{p}'}, \hat{a}_{\bar{p}}] + [\hat{a}^{+}_{\bar{p}'} \hat{a}_{\bar{p}'}, \hat{a}_{\bar{p}}] \right)$$

$$= - E_{\bar{p}}\, \hat{a}_{\bar{p}} \qquad (1.25)$$

$$[\hat{H}, \hat{a}^{+}_{\bar{p}}] = + E_{\bar{p}}\, \hat{a}_{\bar{p}} \qquad (1.26)$$

Thus, if we have an eigenstate of $\hat{H}$, $\hat{H}|\psi\rangle = E_\psi |\psi\rangle$

$$\hat{H}\hat{a}_{\bar{p}}|\psi\rangle = ([\hat{H}, \hat{a}_{\bar{p}}] + \hat{a}_p H)|\psi\rangle = (E_\psi - E_{\bar{p}})|\psi\rangle$$

$$\hat{H}\hat{a}^{+}_{p}|\psi\rangle = (E_\psi + E_{\bar{p}})|\psi\rangle$$

$\hat{a}_{\bar{p}}|\psi\rangle$ and $\hat{a}^{+}_{\bar{p}}|\psi\rangle$ are eigenstates of $\hat{H}$ with energies $E_\psi - E_p$ and $E_\psi + E_{\bar{p}}$!

- Assume : $\exists$ state with minimal energy, vacuum $|0\rangle$: $a_{\bar{p}}|0\rangle = 0 \quad \forall \bar{p}$

$$\langle 0|0\rangle = 1$$

Now, the vectors

$$|\bar{p}\rangle \equiv a^{+}_{\bar{p}}|0\rangle \qquad \text{1-particle states}$$

$$|\bar{p}_1, \dots, \bar{p}_n\rangle \equiv \hat{a}^{+}_{\bar{p}_1}\hat{a}^{+}_{\bar{p}_2}\dots\hat{a}^{+}_{\bar{p}_n}|0\rangle \qquad n\text{-particle states}$$

form the basis of the Fock-space

- Normalization: $\langle \bar{p}|\bar{q}\rangle = \langle 0|\hat{a}_{\bar{p}}\hat{a}^{+}_{\bar{q}}|0\rangle$

$$= \langle 0|[\hat{a}_{\bar{p}}, \hat{a}^{+}_{\bar{q}}] + \hat{a}^{+}_{\bar{q}}\underbrace{\hat{a}_{p}|0\rangle}_{=0} = \delta^{(3)}(\bar{p}-\bar{q}) \qquad (1.27)$$

and likewise for n-particle states

(NOTE: sometimes $\langle \bar{p}|\bar{q}\rangle = 2E_{\bar{p}}(2\pi)^3\delta^{(3)}(\bar{p}-\bar{q})$ is used! This comes from different normalization for $\hat{a}, \hat{a}^{+}$.)

- What is the energy of vacuum ?

$$\langle 0|\hat{H}|0\rangle = \int d^3\bar{p}\, E_{\bar{p}} \langle 0|\hat{a}^{+}_{\bar{p}}\hat{a}_{p} + \tfrac{1}{2}\underbrace{[\hat{a}_{\bar{p}}, \hat{a}^{+}_{\bar{p}}]}_{\delta^{(3)}(\bar{0})}|0\rangle$$

$$= \int d^3\bar{p}\, E_{\bar{p}}\, \tfrac{1}{2}\,\delta^{(3)}(\bar{0}) = \infty\ ! \qquad (1.28)$$

Vacuum energy is infinite !?

This is <u>not</u> a problem, because the vacuum energy is not observable (if we neglect gravity, i.e. GR)

This is related to the 0-point energy of the harmonic oscillator ($E_n = n + \frac{1}{2}$): ∞ number of oscillators $\Rightarrow$ ∞ 0-point energy.

We can here just neglect the infinity.
Nevertheless, this is often "solved" by the "<u>normal ordering</u>" trick:

$:\hat{O}: \equiv \hat{O}$ where all $\hat{a}_{\bar{p}}$'s moved to right of $\hat{a}^{+}_{\bar{p}}$'s.

$$:\hat{H}: = \int d^3\bar{p}\, E_{\bar{p}}\, a^{+}_{\bar{p}} a_{\bar{p}} \quad , \quad :\hat{H}:|0\rangle = 0|0\rangle$$

In this case

$$:\hat{H}:|\bar{p}\rangle = \int d^3\bar{p}'\, E_{\bar{p}'}\, \hat{a}^{+}_{\bar{p}'} \underbrace{\hat{a}_{\bar{p}'}\, \hat{a}^{+}_{\bar{p}}}_{[\hat{a}_{\bar{p}'}, \hat{a}^{+}_{\bar{p}}] + \hat{a}^{+}_{\bar{p}} \hat{a}_{\bar{p}'}} |0\rangle$$

$$= E_{\bar{p}} |\bar{p}\rangle \qquad (1.29)$$

as one might expect.

<u>Note:</u> in cosmology, vacuum energy is very significant — <u>cosmological constant</u>, "dark energy"

The value is all wrong:

$$\epsilon = \int^{\Lambda} d^3\bar{p}\, E_{\bar{p}} \simeq \int^{\Lambda} d^3\bar{p}\, \bar{p} \simeq \Lambda^4 \quad , \ \Lambda \text{ cut-off}$$

natural value w. gravity $\Lambda = M_{Planck} = \sqrt{\frac{\hbar c}{G}} \simeq 10^{19}\,\text{GeV}$

$\Rightarrow \epsilon \sim 10^{76}\,\text{GeV}^4$

Observed (cosmology): $\epsilon \sim 10^{-44}\,\text{GeV}^4$

Estimate incorrect by 10^{-120} ! It is not known how to obtain the correct value

Above we had real field $\varphi(x)$.

In the case of complex field $\phi(x)$, the field operator has 2 independent terms:

$$\hat{\phi}(x) = \int_{\bar{k}} \left[\hat{a}_{\bar{k}} e^{-ik\cdot x} + \hat{b}^{+}_{\bar{k}} e^{ik\cdot x} \right] \qquad (1.30)$$

Here

$$[\hat{a}_{\bar{k}}, \hat{a}^{+}_{\bar{q}}] = [\hat{b}_{\bar{k}}, \hat{b}^{+}_{\bar{q}}] = \delta^{(3)}(\bar{k}-\bar{q}) \qquad (1.31)$$

other commutators $= 0$.

Defining $\hat{\pi} = \partial_0 \hat{\phi}$, we obtain the canonical commutation relations

$$[\hat{\phi}(x^0,\bar{x}), \hat{\pi}(x^0,\bar{y})] = [\hat{\phi}^{+}(x^0,\bar{x}), \hat{\pi}^{+}(x^0,\bar{y})] = i\delta^{(3)}(\bar{x}-\bar{y})$$

Interpretation: consider

$$\hat{\phi}^{+}(x) = \int_{\bar{k}} [\hat{a}^{+}_{\bar{k}} e^{ik\cdot x} + \hat{b}_{\bar{k}} e^{-ikx}]$$

$\hat{a}^{+}_{\bar{k}}$ creates (positive freq.) particle

$\hat{b}_{\bar{k}}$ destroys antiparticle

$$\left.\begin{array}{ll} \hat{a}_{\bar{k}} & \text{destroys particle} \\ \hat{b}^{+}_{\bar{k}} & \text{creates antiparticle} \end{array}\right\} \text{ in } \hat{\phi}(x)$$

Particles: Noether charge +1

Antiparticles: charge −1

Total charge conserved! (Cons. Noether current)

For real field, no conserved charge.

1.4. Propagators

Propagators or 2-point Green functions have a central role in QFTs.

For real-valued field, define Wightman-function

$$W(x,y) = \langle 0 | \hat{\varphi}(x) \, \hat{\varphi}(y) | 0 \rangle \tag{1.32}$$

If $x^0 = y^0$, $W(x,y) = W(y,x)$, but in general not!