Similarly, 4-pt. function to $O(\lambda^2)$

$$\langle \varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)\rangle = \frac{\delta}{i\delta J(x_1)}\cdots\frac{\delta}{i\delta J(x_4)}\left.\frac{Z[J]}{Z[0]}\right|_{J=0} =$$

$$= \prod_{i=1}^{4}\frac{\delta}{i\delta J(x_i)}\left(1 - i\frac{\lambda}{4!}\int d^4x\left[-6G_{xx}(G_{xy}J_y)^2 + (G_{xy}J_y)^4\right]\right)\left.\frac{Z_0[J]}{Z_0[0]}\right|_{J=0}$$

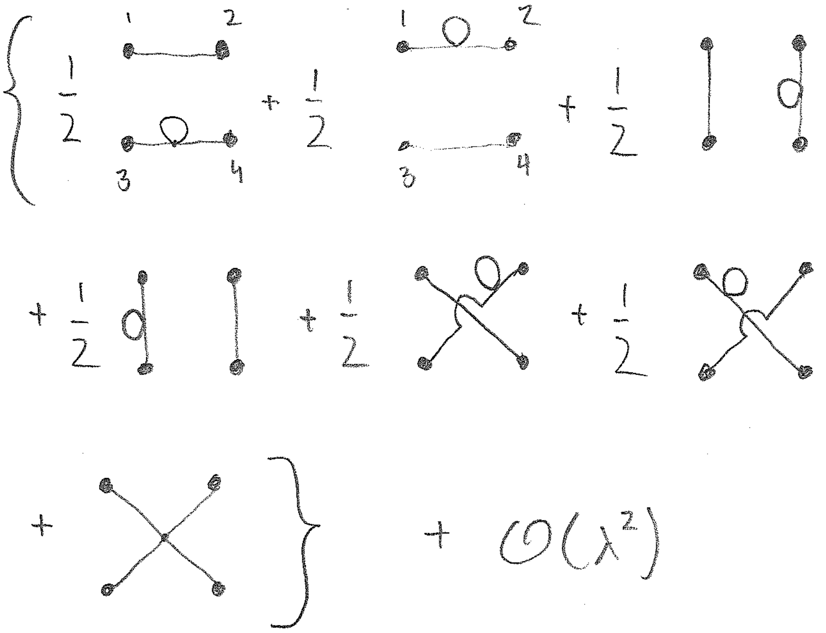
gives $O(\lambda^0)$, (2.28), omit this here

requires all 4 δ's

requires 2 δ's, thus, 2 go to where $\frac{\delta}{i\delta J_1}\frac{\delta}{i\delta J_2} = G_{12}$

$= (2.28) +$

$-i\lambda\Big\{$



$\Big\} + O(\lambda^2)$ (2.39)

$= -i\lambda\int d^4x\, G_{xx_1}G_{xx_2}G_{xx_3}G_{xx_4}$

Generating functionals:

A) $Z[J]$: $\left(\frac{\delta}{i\delta J}\right)^n Z[J]\Big|_{J=0}$ generates all diagrams with n external points, including "vacuum" contributions not connected to any external points (cf. (2.38))

B) $\frac{Z[J]}{Z[0]}$: $\left(\frac{\delta}{i\delta J}\right)^n \frac{Z[J]}{Z[0]}\Big|_{J=0}$ all real n-point diagrams, where all parts are connected to (at least) one external point (often denoted by $\mathcal{Z}[J]$)

C) $W[J] \equiv \ln Z[J] \Leftrightarrow Z[J] = e^{W[J]}$ (2.40)

$\left(\frac{\delta}{i\delta J}\right)^n W[J]\Big|_{J=0}$ generates all connected n-point diagrams! (no proof here) (sometimes with an extra i: $Z = e^{iW}$)

$$G_c^{(n)}(x_1 .. x_n) \equiv \langle 0 | T\hat{\varphi}(x_1) .. \hat{\varphi}(x_n) | 0 \rangle_c = \frac{\delta}{i\delta J(x_1)} \cdots \frac{\delta}{i\delta J(x_n)} W[J]\Big|_{J=0} \tag{2.41}$$

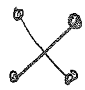
Let's check it:

$$\left.\frac{\delta^2 W}{i\delta J(x_1)\, i\delta J(x_2)}\right|_{J=0} = \frac{1}{Z}\frac{\delta^2 Z}{i\delta J(x_1)\, i\delta J(x_2)} - \frac{1}{Z^2}\underbrace{\cancel{\left.\frac{\delta Z}{i\delta J(x_1)}\right|_{J=0}}}_{0} \cdot \underbrace{\left.\frac{\delta Z}{i\delta J(x_2)}\right|_{J=0}}_{0}$$

$$= G_F(x_1 - x_2) =$$  OK

$$\frac{\delta^4 W}{i\delta J(x_1) \cdots i\delta J(x_4)} = \frac{-1}{Z^2}\frac{\delta^2 Z}{i\delta J_1\, i\delta J_2}\frac{\delta^2 Z}{i\delta J_3\, i\delta J_4} + 2 \text{ perm.}$$

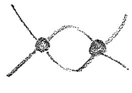
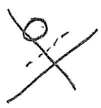
$$+ \frac{1}{Z}\frac{\delta^4 Z}{i\delta J_1 \cdots i\delta J_4}$$

The terms on 1st line give exactly those <u>disconnected</u> terms arising from the 2nd line term, (2.39), but with opposite sign. The only term which remains is .

D) $\underline{\Gamma[J] \equiv W[J] - \int d^4x\, \phi(x) J(x)}$ (Legendre transformation)

where $\phi(x) \equiv \dfrac{\delta W[J]}{\delta J(x)}$ (2.42)

Generates 1-particle-irreducible (1PI) diagrams, i.e. diagrams which cannot be made disconnected by cutting <u>one</u> internal line

 is 1PI  is <u>not</u> 1PI

Amputed Green functions

Often we need Green functions where the external legs (Feynman propagators) have been removed:

$$G_{\tilde{c}}^{(n)}(x_1 .. x_n) = \langle 0|T\hat{\varphi}(x_1) .. \hat{\varphi}(x_n)|0\rangle_{\tilde{c}} \tag{2.43}$$

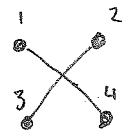
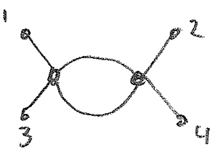
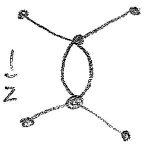
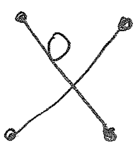
$$= \int d^4y_1 .. d^4y_n\, G_C^{(n)}(y_1 .. y_n)\, G_F^{-1}(y_1 - x_1) .. G_F^{-1}(y_n - x_n)$$

Here $\int d^4y\, G_F(x-y)\, G_F^{-1}(y-z) = \delta^4(x-z)$

Feynman rules

The connected Green function is obtained from taking derivatives of W: for example,

$$G_C^4(x_1 .. x_4) = \left.\frac{\delta^4 W}{i\delta J(x_1) .. i\delta J(x_4)}\right|_{J=0} \tag{2.44}$$

$=$  $+ \frac{1}{2}$  $+ \frac{1}{2}$  $+ \frac{1}{4}$ 

$O(\lambda)$ $\quad$ $O(\lambda^2)$ $\quad$ $+3$ perm.

$+ O(\lambda^3)$

It is relatively easy to identify the right diagrams, but how to get the numerical

Coefficients in front ?

Feynman rules codify this (in coordinate space)

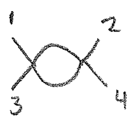
1. propagators  $= G_F(x-y)$
2. vertices  $= (-i\lambda)\int d^4z$
3. External point  $= 1$
4. Multiply the diagram with the <u>symmetry factor</u> s

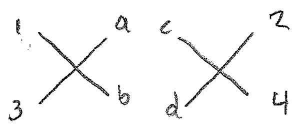
(2.45)

For example, diagram 2 in (2.44) is

$$s \times (-i\lambda)^2 \int d^4y_1\, d^4y_2 \left(G_F(y_1-y_2)\right)^2 G_F(x_1-y_1)\, G_F(x_3-y_1) \times G_F(x_2-y_2)\, G_F(x_4-y_2)$$

The <u>symmetry factor</u> s counts the number of equivalent ways to contract the points, or in how many ways $\frac{\delta}{\delta J}$ can act on $W[J]$ and produce the same diagram.

<u>Calculation rule</u>: example 

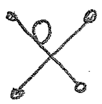
- take the vertices separately: 
- x_1 can be connected to any of 8 available legs
- x_3 has 3 options (same vertex!)
- x_2 has 4 options
- x_4 has 3
- 2 options to connect vertices, a-c, b-d or a-d, b-c

- both vertices give $\frac{1}{4!}$ $\left(\frac{1}{4!}\lambda\varphi^4\right)$
- $\exp(iS_I) = 1 + iS_I + \frac{(i)^2}{2!}S_I^2 + \ldots$
  was expanded to 2nd order to get λ^2.
  Thus, we get extra $\frac{1}{2!}$

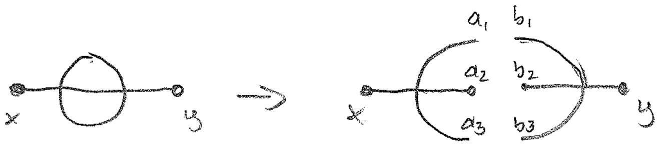
$\Rightarrow$ thus, overall $S = \frac{8\cdot3\cdot4\cdot3\cdot2}{(4!)^2\cdot 2!} = \frac{1}{2}$.

Examples:

$S = \frac{4\cdot3\cdot2\cdot1}{4!\ 1!} = 1$ as we got before



$S = \frac{4\cdot4\cdot3\cdot2\cdot3}{(4!)^2\cdot 2!} = \frac{1}{4}$



x y $a_1 \to b$ a_2 a_1

$S = \frac{4\cdot4\cdot3\cdot2\cdot1}{(4!)^2\cdot 2!} = \frac{1}{6}$

$S = \frac{3\cdot1}{4!\,1!} = \frac{1}{8}$

Choose any leg, 3 options to connect to another.

- The factor $\frac{1}{4!}$ could as well be included in 2. However, this way is more customary. As we see, 4! is largely cancelled!

- For Amputated Green functions the external propagators are substituted by δ-functions (2.43); $G_F(x-y) \to \delta^{(4)}(x-y)$

For example

amputed

$$-i\lambda \int d^4z \; G_F(x_1-z) G_F(x_2-z) G_F(x_3-z) G_F(x_4-z)$$

$$\rightarrow (-i\lambda) \int d^4z \; \delta^{(4)}(x_1-z)\, \delta^{(4)}(x_2-z)\, \delta^{(4)}(x_3-z)\, \delta^{(4)}(x_4-z)$$

$$\left(\int d^4y \; G_F(x-y)\, G_F^{-1}(y-z) = \delta^{(4)}(x-z) \right)$$

Momentum space amputed Green functions

- These are the standard tool used in computations!

- Recall:

$$\begin{cases} G_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2-m^2+i\epsilon} e^{-ip\cdot(x-y)} \\ \tilde{G}_F(p) = \frac{i}{p^2-m^2+i\epsilon} \end{cases} \tag{2.46}$$

- Vertex in mom. space: take F.T. over external x_i:

$$\int d^4x_1 \ldots d^4x_4 \; e^{ip_1\cdot x_1} \times \ldots \times e^{ip_4\cdot x_4} \underbrace{\left[(i\lambda) \int d^4z \; \delta^{(4)}(x_1-z) \ldots \delta^{(4)}(x_4-z) \right]}_{\text{vertex = amputed 4-pt. Green function to order } \lambda}$$

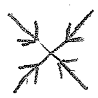
$$= \underline{(-i\lambda)\cdot(2\pi)^4\, \delta^{(4)}(p_1+p_2+p_3+p_4)} \tag{2.47}$$

Thus, n-point connected amputed Green functions in mom. space is

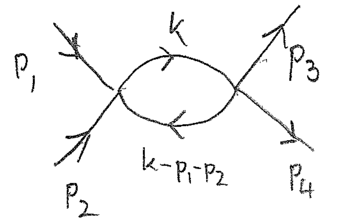
$$\tilde{G}_{\tilde{c}}(p_1 \dots p_n) \equiv \int d^4x_1 \dots d^4x_n \, e^{ip_1\cdot x_1} \dots e^{ip_n\cdot x_n} \, G_{\tilde{c}}(x_1 \dots x_n)$$

- Every external leg has momentum p_i
- Internal propagators $G(x-y) \to \int \frac{d^4p}{(2\pi)^4} \tilde{G}(p) e^{-ip\cdot(x-y)}$
- Integrals over coordinates give, for each vertex, $(-i\lambda)(2\pi)^4 \delta^{(4)}(\sum_i p_i)$
- δ-functions at vertices cancel p-integrals so that:
  - overall $(2\pi)^4 \delta^{(4)}(\sum_{\substack{\text{external}\\ p_i}} p_i)$ remains
  - for each closed loop there remains integral $\int \frac{d^4p}{(2\pi)^4}$.

This gives momentum space Feynman rules for amputed Green functions:

1. Internal propagator $\xrightarrow{p}$ $= \frac{i}{p^2-m^2+i\epsilon}$
2. Vertex  $= -i\lambda$,
   with 4-momentum conservation $p_1+p_2+p_3+p_4=0$ 
3. Closed loop $\int \frac{d^4k}{(2\pi)^4}$ (2.48)
4. Total 4-mom. conservation $(2\pi)^4 \delta^{(4)}(\sum_{\text{external}} p_i)$
5. Symmetry factor

As a concrete example, diagram



note: different direction for p_3, p_4!

should be $\frac{1}{2}(2\pi)^4 \delta^{(4)}(p_1+p_2-p_3-p_4)\cdot(-i\lambda)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-m^2+i\epsilon} \frac{i}{(k-p_1-p_2)^2-m^2+i\epsilon}$

↖ s, page 67

Let us verify this is the case!

$$\int d^4x_1 \cdots d^4x_4 \, G^{(4)}(x_1 \ldots x_4) e^{ip_1\cdot x_1} \cdots e^{ip_4 \cdot x_4}$$

$$= \frac{1}{2}(-i\lambda)^2 \int d^4x_1 \cdots d^4x_4 \, e^{ip_1\cdot x_1 + ip_2\cdot x_2 - ip_3\cdot x_3 - ip_4\cdot x_4} \times$$

$$\int d^4y_1 \, d^4y_2 \, (G_F(y_1-y_2))^2 \delta^{(4)}(x_1-y_1)\delta^{(4)}(x_3-y_1)\delta^{(4)}(x_2-y_2)\delta^{(4)}(x_4-y_2)$$

$$= \frac{1}{2}(-i\lambda)^2 \int d^4y_1 \, d^4y_2 \, e^{iy_1(p_1+p_2) - iy_2\cdot(p_3+p_4)} \int \frac{d^4k_1 \, d^4k_2}{(2\pi)^8} \frac{i}{k_1^2-m^2} \frac{i}{k_2^2-m^2}$$

$$\times e^{-ik_1\cdot(y_1-y_2) - ik_2\cdot(y_1-y_2)}$$

$$= \frac{1}{2}(-i\lambda)^2 \int dk_1 \, dk_2 \, \delta^{(4)}(k_1+k_2-p_1-p_2)\, \delta^{(4)}(k_1+k_2-p_3-p_4) \frac{i}{k_1^2-m^2} \frac{i}{k_2^2-m^2}$$

$$= \frac{1}{2}(-i\lambda)^2 \cdot (2\pi)^4 \delta^{(4)}(p_1+p_2-p_3-p_4) \cdot \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-m^2+i\epsilon} \frac{i}{(k-p_1-p_2)^2-m^2+i\epsilon}$$

which was the desired result!

Feynman rules seem to work.