Relation to scattering

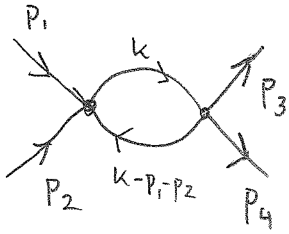
We will not discuss scattering problems in this course; see, for example, Introduction to particle physics. Here we just note that the "amplitude" $\mathcal{M}$ there is just $\tilde{G}_c^{(n)}$, amputated n-point Green function in p-space.

(see particle physics notes)

3. Renormalization

3.1 Divergences

- Consider again the $\mathcal{O}(\lambda^2)$ correction on page 70 to 4-pt. function:



Drop $(2\pi)^4\,\delta^{(4)}(\Sigma p_i)$, as is often done. Total 4-mom. conservation is implicitly there

- Now the amputated Green function = Amplitude is

$$A = (-i\lambda)^2 \cdot \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\,\frac{i}{k^2-m^2+i\varepsilon}\,\frac{i}{(k-p_1-p_2)^2-m^2+i\varepsilon} \tag{3.1}$$

- What is the value? When $k\to\infty$,

$$A \propto \int^{\Lambda} d|k|\,\frac{1}{|k|} \sim \ln\Lambda\,, \quad \Lambda \text{ "cut-off"}$$

- Thus, diagram <u>diverges logarithmically</u> as $k\to\infty$!

- Or;  loop $\rightarrow \int \frac{d^4k}{(2\pi)^2}\,\frac{1}{k^2-m^2} \sim \int^{\Lambda} d|k|\cdot|k|$

$$\sim \Lambda^2$$

<u>Diverges quadratically</u> as $k\to\infty$!

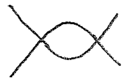
- We see that the divergences appear as $\underline{k \to \infty}$, or $\underline{x-y \to 0}$ in coord. space

  $\to$ Ultraviolet divergences

- These infinities plague QFT's in general, and cause serious headache! we have apparently ∞ quantities, e.g.

$$G_c^{(4)} = \quad \text{[diagram 1]} \quad + \quad \text{[diagram 2]} \quad + \dots$$

λ



$\lambda^2 \times \infty$

- All is not lost, however: physically we observe $G_c^{(4)}$ (or scattering cross-section derived from it), not bare λ.

- In renormalizable theories we can absorb the infinities by redefinition of the parameters of the theory. In this case

$$\lambda \to \lambda + \delta\lambda \quad ; \quad \delta\lambda = -\lambda \times \infty$$

so that ∞'s cancel! $\Rightarrow$ Renormalization

- In order to control this, we need to quantify the infinities $\Rightarrow$

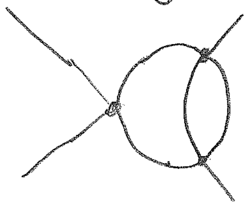
  Regularization

Central issues with QFT's !!

General notes about the divergences:

(in perturbation theory!)

General diagram:



V vertices

E external legs

I internal lines

L loops

in d dimensions ($d=4$ for us)

- Each loop $\sim \int \frac{d^d p}{(2\pi)^d} \sim p^d$ at large p
- internal line $\sim \frac{1}{p^2}$

$\Rightarrow$ Apparent divergence $=$ $D = dL - 2I$ (3.2)

- We can relate L to other quantities:
  - Each vertex has 4 legs, each internal line takes 2 of these:

    $$E + 2I = 4L \tag{3.3}$$

  - Each internal line brings, in principle, $\int d^d p$, but momentum conservation at each vertex kills most of these; what remain are the loop integrals and total mom. conservation

    $$L = I - V + 1 \tag{3.4}$$

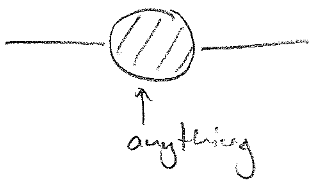
    total mom. conservation

Thus, $\quad I = \frac{1}{2}(4V - E)$

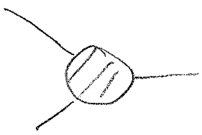
and
$$D = d\left(\tfrac{1}{2}(4V-E) - V + 1\right) - 2\tfrac{1}{2}(4V-E)$$
$$= d + (d-4)V - \tfrac{1}{2}(d-2)E \qquad (3.5)$$

We see:

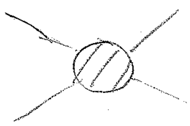
* If $d < 4$, D becomes smaller the larger V is $\Rightarrow$ $\exists$ divergences only up to some (low) loop order (V, L)
* If $d > 4$, divergences become worse as the loop order increases.
* $d = 4$: degree of divergence only depends on E:



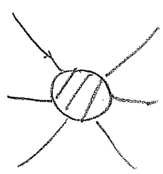
$D = 2$ quadratic



cannot appear!

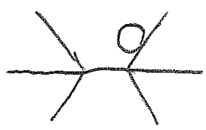


$D = 0 \to$ log-divergence



$D = -2$ not divergent (?)

However, even if $D<0$, divergences can appear in subdiagrams:



divergent ! (of type $D=2$)

NOTE: the above was for $\lambda \varphi^4$ - interaction, with 4 legs/vertex. If each vertex has N legs ($\lambda\varphi^N$), (3.2) and (3.4) remain, but (3.3) becomes $E+2I = N\cdot L$

$$\Rightarrow D = d - \tfrac{1}{2}(d-2)E + \left(\tfrac{1}{2}(d-2)N - d\right)V \qquad (3.6)$$

If, for example, $N=6$ ($\lambda\varphi^6$), the last term ($d=4$) is $2V \Rightarrow$ divergences become more serious as number of vertices V grows

$\Rightarrow$ theory is not renormalizable. (OK if $d\leq 3$!)

IDEA OF RENORMALIZATION:

- Can we "hide" the divergences into $\frac{1}{2}m^2\varphi^2$ and $\frac{1}{4!}\lambda\varphi^4$, by changing m^2 and λ ?

3.2 Regularization

In order to control the infinities we need to regularize them somehow.

There are various ways to do this:

1. Lattice regularization:

   When we derived the path integral, we discretized space and time and let the interval $\to 0$. However, if we keep discretization, the UV divergences become finite!

   - Mathematically the best defined method
   - Computer simulations
   - Difficult to treat in analytical computations

2. Momentum regularization

   - In $\int d^4k \to \int^{|k|<\Lambda} d^4k$, Λ momentum cutoff
   - At the end, let $\Lambda \to \infty$
   - Conceptually simple, difficult in practice (if $|p|<\Lambda$, $|q|<\Lambda$, what is $|p-q|$?)
   - Difficult to satisfy symmetries (gauge theory etc.)

3. Pauli-Villars regularization, "proper time" regularization by Schwinger, ...

4 Dimensional regularization

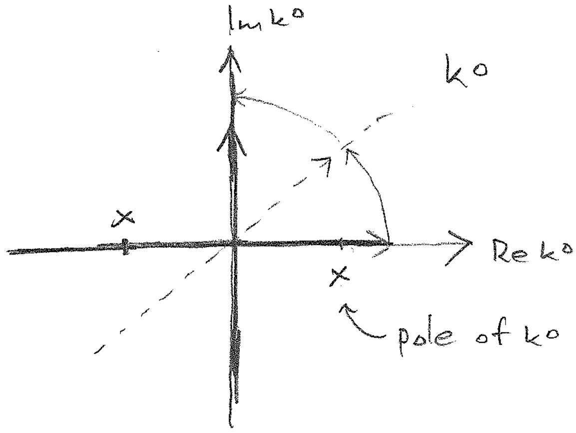
- Pure magic: calculate in $4-2\epsilon$ dimensions, and divergences disappear! ('t Hooft 1972) (appear as $1/\epsilon$)

- We shall discuss this method
- standard tool

Euclidization:

- Before going to continuous dimensions, it is convenient to get rid of the non-uniform metric $g = \mathrm{diag}(1,-1,-1,-1)$.
- Consider Feynman propagator (where k integration variable)

$$\frac{i}{k^2 - m^2 + i\epsilon} = \frac{i}{k_0^2 - E_k^2 + i\epsilon}$$



- In complex k^0-plane:

We can rotate the integration path of k^0,

$$\begin{cases} k^0 \to i k^0 \\ dk^0 \to i\, dk^0 \end{cases} \qquad \vec{k} \to \vec{k}$$

without encountering any poles.

- Thus, if the contributions of the "arches" at $|k^0| \to \infty$ vanish, (as they do) the value of the integral does not change

- Thus, denoting rotated k by k_E, (Euclidean)

$$\int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-m^2+i\epsilon} = i\int \frac{d^4k_E}{(2\pi)^4} \frac{i}{-k_E^2-m^2} = i\int \frac{d^4k_E}{(2\pi)^4} \frac{-i}{k_E^2+m^2}$$

where $k_E^2 = k_E^{0\,2} + \bar{k}_E^{\,2}$ (3.7)

- This can be done to all internal & external momenta of a diagram. At the end we can turn ext. momenta → Minkowski

Here $\frac{i}{(k+p+q)^2-m^2+i\epsilon} \rightarrow \frac{1}{(k_E+p_E+q_E)^2+m^2}$

↑ any combination

- Euclidean Feynman propagator: $\frac{1}{k^2+m^2}$; $k^2 = k^{0\,2}+\bar{k}^2$

no need for $i\epsilon$! (3.8)

- Note: we could have defined everything in Euclidean space from the beginning, including $\mathcal{L}$ and S. Here we use this just to evaluate the momentum integrals in the diagrams

3.3. Dimensional regularization

- Consider space-time dimensionality d as a complex number! Usually $d = 4 - 2\epsilon$. At the end use analytic continuation to $d \to 4$. Divergences appear as poles $\frac{1}{d-4} \sim \frac{1}{\epsilon}$

- For concreteness, use Euclidized momenta:

$$\left\{ \begin{aligned} & \int_p \equiv \int \frac{d^4 p}{(2\pi)^4} \to \int \frac{d^d p}{(2\pi)^d} \\ & \int d^4 x \to \int d^d x \\ & (2\pi)^4 \delta^{(4)}(p) \longrightarrow (2\pi)^d \delta^{(d)}(p) \\ & \int d^d x \, e^{ip\cdot x} = (2\pi)^d \delta^{(d)}(p) \end{aligned} \right. \tag{3.9}$$

- These still obey the standard "Axioms":

$$\begin{aligned} &(a) \int d^d p \, f(p+q) = \int d^d p \, f(p) \\ &(b) \int d^d p \, f(\lambda p) = |\lambda|^{-d} \int d^d p \, f(p) \\ &(c) \int d^d p \int d^{d'} q \, f(p) \, g(q) = \int d^d p \, f(p) \cdot \int d^{d'} q \, g(q) \end{aligned} \tag{3.10}$$

Surface area of d-dim. sphere is often needed:

$$\text{Now} \int d^d r \, e^{-ar^2} \underset{(c)}{=} \left[\int_{-\infty}^{\infty} dr_1 \, e^{-ar_1^2}\right]^d = \left(\frac{\pi}{a}\right)^{d/2}$$

On the other hand,

$$\int d^d r \, e^{-ar^2} = C(d) \int_0^\infty d\rho \, \rho^{d-1} e^{-a\rho^2}$$

$C(d)$: Area of d-dim. sphere of $r=1$

$$= C(d) \, a^{-d/2} \int_0^\infty dx \, x^{d-1} e^{-x^2}$$

$y = x^2$
$dy = 2x\,dx$

$$= \frac{C(d)}{2a^{d/2}} \underbrace{\int_0^\infty dy \, y^{\frac{d}{2}-1} e^{-y}}_{=\Gamma\left(\frac{d}{2}\right)}$$

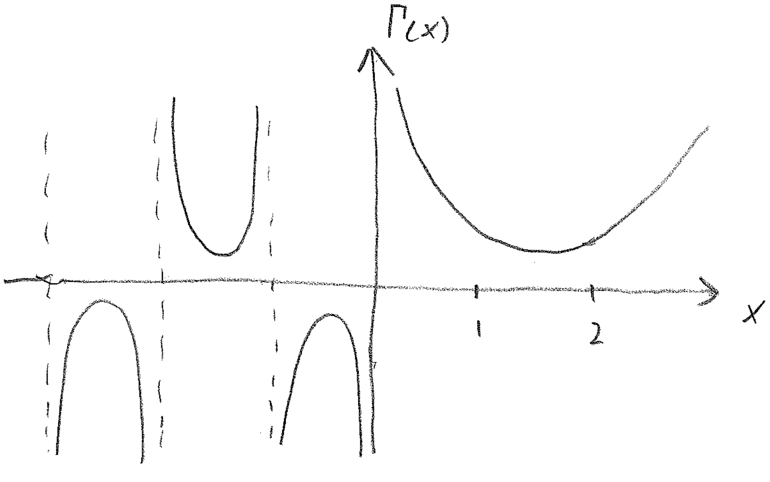
$\Gamma(n) = (n-1)! = (n-1)\cdot(n-2)\cdot \ldots \cdot 1$

$$\Rightarrow C(d) = \frac{2\pi^{d/2}}{\Gamma(d/2)} \tag{3.11}$$

$(C(2) = 2\pi, \; C(3) = 4\pi)$

$$\text{Note: } \Gamma\left(\tfrac{1}{2}\right) = \int_0^\infty dy \, y^{-1/2} e^{-y} = 2\int_0^\infty dx \, e^{-x^2} = \sqrt{\pi} \tag{3.12}$$

Γ-function has poles at $0, -1, -2, \ldots$



- $\Gamma(1+\epsilon) = \Gamma(1) + \epsilon\Gamma'(1) + \mathcal{O}(\epsilon^2) = 1 - \gamma_E \cdot \epsilon + \mathcal{O}(\epsilon^2)$ (3.13)

Euler-gamma
(Euler-Mascheroni)
$\gamma_E = 0.577215665\ldots$

Because $\Gamma(x+1) = x\Gamma(x) \Rightarrow$ (3.14)

- $\Gamma(\epsilon) = \frac{1}{\epsilon}\Gamma(1+\epsilon) = \frac{1}{\epsilon} - \gamma_E + \mathcal{O}(\epsilon)$ (3.15)

- $\Gamma(-1+\epsilon) = \frac{-1}{\epsilon-1}\Gamma(\epsilon) = -\frac{1}{\epsilon}\frac{1}{1-\epsilon}(1-\gamma_E\epsilon + \mathcal{O}(\epsilon^2))$ (3.16)

$= -\frac{1}{\epsilon}(1+(1-\gamma_E)\epsilon + \mathcal{O}(\epsilon^2))$

Note also

- $a^\epsilon = e^{\epsilon \ln a} = 1 + \epsilon \ln a + \mathcal{O}(\epsilon^2)$ (3.17)

Peculiar consequence of d-dim. integral rules:

$$\int d^d p = 0 = \int d^d p \cdot \frac{1}{(p^2)^n}, \quad n \neq d/2 \qquad (3.18)$$

because of (b)! $\left(\int d^d p = |\lambda|^{-d} \int d^d p \right)$

This looks crazy; according to more normal rules these are ∞! But this works.

We can also easily derive (p: euclidean!)

$$\int d^d p \, \frac{1}{p^2+m^2} = \frac{\pi^{d/2}}{[m^2]^{1-d/2}} \Gamma\left(1-\frac{d}{2}\right) \qquad (3.19)$$

or its generalization: (derivate (3.19) wrt. m^2)

$$\int d^d p \, \frac{1}{[p^2+m^2]^n} = \frac{\pi^{d/2}}{[m^2]^{n-d/2}} \frac{\Gamma(n-d/2)}{\Gamma(n)} \qquad (3.20)$$

Many other integration rules can be derived!

Regularization of 2-pt. function

- Consider (2.34) (amputated!)

$$= -i\frac{\lambda}{2}\int \frac{d^4 p}{(2\pi)^4} \frac{i}{p^2-m^2+i\epsilon}$$

- Euclideanize, d-dim, use (3.19):

$$\rightarrow -i\frac{\lambda}{2}\int \frac{d^d p}{(2\pi)^d} \frac{1}{p^2+m^2} = -i\frac{\lambda}{2} \frac{\Gamma(1-d/2)}{(4\pi)^{d/2}(m^2)^{1-d/2}}$$

- set $d = 4-2\epsilon$, use (3.16)

$$= +i\frac{\lambda}{2}\frac{(m^2)^{1-\epsilon}}{(4\pi)^{2-\epsilon}}\left(\frac{1}{\epsilon}+1-\gamma_E+O(\epsilon)\right) \qquad (3.21)$$

divergent as $\epsilon \to 0$
($d \to 4$)

The divergent part is

$$= i\frac{\lambda}{2}\frac{m^2}{(4\pi)^2}\frac{1}{\epsilon} + \mathcal{O}(1) \qquad (3.22)$$

In order to get $\mathcal{O}(1)$ part, use (3.17) to

$$= i\frac{\lambda}{2}\frac{m^2}{(4\pi)^2}\left(\frac{m^2}{4\pi}\right)^{-\epsilon}\left(\frac{1}{\epsilon}+1-\gamma_E+\mathcal{O}(\epsilon)\right)$$

- Now $a^\epsilon = 1+\epsilon\ln a + \mathcal{O}(\epsilon^2)$, but we really should not take log of a dimensionful quantity (m^2).
- Introduce new mass scale μ, and write

$$\underline{(m^2)^{-\epsilon}} = \mu^{-2\epsilon}\left(\frac{m^2}{\mu^2}\right)^{-\epsilon} = \underline{\mu^{-2\epsilon}\left(1+\epsilon\ln\frac{\mu^2}{m^2}+\mathcal{O}(\epsilon^2)\right)} \qquad (3.23)$$

At this stage μ is unphysical & artificial, but it will have an important role later! So far it is unspecified.

- $$(3.21) = i\frac{\lambda}{2}\frac{m^2}{(4\pi)^2}\mu^{-2\epsilon}\left(\frac{1}{\epsilon}+1-\gamma_E+\ln\frac{4\pi\mu^2}{m^2}+\mathcal{O}(\epsilon)\right) \qquad (3.24)$$

What to do with the divergent piece?