Fermionic systems

Electron gas

The ideal Fermi gas is a good approximation for example for the conducting electrons in a metal.
When the single particle energies are $\epsilon_k = \frac{\hbar^2 k^2}{2m}$ is

$$\begin{aligned}\omega_1(\epsilon) &= V\,\frac{g_s}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\sqrt{\epsilon} \\ &= V 2\pi g_s \left(\frac{2m}{h^2}\right)^{3/2}\sqrt{\epsilon}.\end{aligned}$$

As the density we get

$$\rho = \frac{\bar{N}}{V} = \frac{g_s}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\int_0^\infty d\epsilon\,\frac{\sqrt{\epsilon}}{e^{\beta(\epsilon-\mu)}+1}.$$

The energy per particle will be

$$\bar{\epsilon} = \frac{E}{\bar{N}} = \frac{\int_0^\infty d\epsilon\,\frac{\epsilon^{3/2}}{e^{\beta(\epsilon-\mu)}+1}}{\int_0^\infty d\epsilon\,\frac{\epsilon^{1/2}}{e^{\beta(\epsilon-\mu)}+1}}.$$

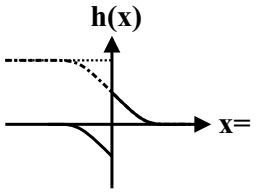
Degenerated Fermi gas
Suppose that $k_B T \ll \mu$.
Let's write

$$\frac{1}{e^{\beta(\epsilon-\mu)}+1} = \theta(\mu-\epsilon) + h(\epsilon-\mu),$$

where

$$h(x) = \mathrm{sign}(x)\,\frac{1}{e^{\beta|x|}+1}.$$

The function $h(x)$ deviates from zero only at the narrow domain $|x| \lesssim k_B T \ll \mu$.



Let's evaluate the integral

$$\begin{aligned}&\int_0^\infty d\epsilon\,\frac{\phi(\epsilon)}{e^{\beta(\epsilon-\mu)}+1} \\ &\quad= \int_0^\infty d\epsilon\,\phi(\epsilon)\,[\theta(\mu-\epsilon)+h(\epsilon-\mu)] \\ &\quad= \int_0^\mu d\epsilon\,\phi(\epsilon) + \int_0^\infty d\epsilon\,h(\epsilon)\,[\phi(\mu+\epsilon)-\phi(\mu-\epsilon)] \\ &\qquad+\int_\mu^\infty d\epsilon\,h(\epsilon)\phi(\mu-\epsilon).\end{aligned}$$

The last term is of the order

$$h(\mu) = \frac{1}{e^{\mu/k_B T}+1} \approx e^{-\mu/k_B T}$$

and can be discarded.
If $\phi(\epsilon)$ regular enough in the vicinity of $\epsilon \approx \mu$ we can expand

$$\phi(\mu+\epsilon)-\phi(\mu-\epsilon) \approx 2\phi'(\mu)\epsilon + 2\,\frac{1}{3!}\,\phi'''(\mu)\epsilon^3 + \cdots.$$

Now

$$\begin{aligned}&\int_0^\infty d\epsilon\,\frac{\phi(\epsilon)}{e^{\beta(\epsilon-\mu)}+1} \\ &\quad\approx \int_0^\mu d\epsilon\,\phi(\epsilon) \\ &\qquad+2\phi'(\mu)(k_B T)^2\int_0^\infty dz\,\frac{z}{e^z+1} \\ &\qquad+2\phi'''(\mu)\,\frac{1}{3!}\,(k_B T)^4\int_0^\infty dz\,\frac{z^3}{e^z+1} \\ &\qquad+\cdots\end{aligned}$$

and we end up with *Sommerfeld's expansion*

$$\begin{aligned}\int_0^\infty d\epsilon\,\frac{\phi(\epsilon)}{e^{\beta(\epsilon-\mu)}+1} \;\approx\; &\int_0^\mu d\epsilon\,\phi(\epsilon) + \frac{\pi^2}{6}\,(k_B T)^2\phi'(\mu) \\ &+\frac{7\pi^4}{360}\,(k_B T)^4\phi'''(\mu)+\cdots.\end{aligned}$$

Temperature $T = 0$
Now

$$\bar{n}(\epsilon) = \theta(\mu-\epsilon)$$

and $h(x) = 0$.
The *Fermi energy* is

$$\epsilon_F = \mu = \frac{\hbar^2 k_F^2}{2m}.$$

The *Fermi momentum* is

$$p_F = \hbar k_F.$$

The density is

$$\begin{aligned}\rho &= \frac{g_s}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\int_0^\mu d\epsilon\,\sqrt{\epsilon} \\ &= \frac{g_s}{6\pi^2}\left(\frac{2m\epsilon_F}{\hbar^2}\right)^{3/2}\end{aligned}$$

or

$$\rho = \frac{g_s}{6\pi^2}\,k_F^3.$$

The spin degeneracy factor of electrons is $g_s = 2\cdot\frac{1}{2}+1 = 2$, so

$$\rho = \frac{k_F^3}{3\pi^2}.$$

For the energy per particle we get

$$\begin{aligned}\bar{\epsilon} &= \epsilon_F\,\frac{\int_0^1 dx\,x^{3/2}}{\int_0^1 dx\,x^{1/2}} = \epsilon_F\,\frac{2/5}{2/3} \\ &= \frac{3}{5}\,\epsilon_F.\end{aligned}$$

The total energy is

$$E = \frac{3}{5}\,\epsilon_F N = \frac{3}{5}\,N\frac{\hbar^2}{2m}\left(\frac{6\pi^2\rho}{g_s}\right)^{2/3}.$$

Since

$$E = \text{constant} \times N^{5/3} V^{-2/3},$$

we have

$$-p = \left(\frac{\partial E}{\partial V}\right)_N = -\frac{2}{3}\frac{E}{V}$$

or

$$pV = \frac{2}{3}E.$$

Metallic electron gas
When we write the density as

$$\rho = \frac{\bar{N}}{V} = \frac{1}{\frac{4}{3}\pi r_i^3}$$

and define the dimensionles number

$$r_s = \frac{r_i}{a_0},$$

where a_0 is the Bohr radius

$$a_0 = \frac{4\pi\epsilon_0\hbar^2}{me^2} = 0.529\text{Å},$$

we can see that

$$\rho = \frac{1.613 \cdot 10^{30}}{r_s^3}\frac{1}{\text{m}^3}.$$

For metals we have

$$1.9 \lesssim r_s \lesssim 5.6.$$

The Fermi wave vector will become

$$k_F = \frac{1}{a_0 r_s}\sqrt[3]{\frac{9\pi}{4}} = \frac{1.92}{a_0 r_s}.$$

The *Fermi velocity* is

$$\begin{aligned} v_F &= \frac{p_F}{m} = \frac{\hbar k_F}{m} = \frac{1.92\hbar}{m a_0 r_s} \\ &= \frac{4.2 \cdot 10^6}{r_s}\frac{\text{m}}{\text{s}}. \end{aligned}$$

For example in aluminium

$$v_F = 2029\frac{\text{km}}{\text{s}} = \frac{c}{148}.$$

The *Fermi temperature* or the degeneracy temperarure T_F is defined so that

$$k_B T_F = \epsilon_F.$$

Now

$$\epsilon_F = \underbrace{\frac{\hbar^2}{2ma_0^2}}_{\substack{\text{binding} \\ \text{energy of} \\ \text{hydrogen}}} \left(\frac{1.92}{r_s}\right)^2 = \frac{3.69}{r_s^2}\,13.6\text{eV}.$$

Since

$$1\text{eV} = 11604 k_B\text{K},$$

we have

$$T_F = \left(\frac{1.92}{r_s}\right)^2 13.6 \cdot 11604\text{K}.$$

For aluminium the Fermi temperature is $T_F = 136\,000$K.
In general, the metals satisfy

$$T \ll T_F,$$

so the metallic electron gas is strongly degenerated.
Specific heat Let now $T > 0$, but $T \ll T_F$.
We need $\mu = \mu(T)$, when $\frac{\bar{N}}{V} = \rho$ is known.
With the help of the Sommerfeld expansion we get

$$\begin{aligned} \frac{2}{3}\epsilon_F^{3/2} &= \frac{4\pi^2}{g_s}\left(\frac{\hbar^2}{2m}\right)^{3/2}\rho = \int_0^\infty d\epsilon\,\frac{\epsilon^{1/2}}{e^{\beta(\epsilon-\mu)}+1} \\ &\approx \frac{2}{3}\mu^{3/2} + \frac{\pi^2}{12}(k_B T)^2\frac{1}{\sqrt{\mu}} + \cdots \end{aligned}$$

so we can write

$$\frac{2}{3}\mu^{3/2}\left[1 + \frac{\pi^2}{8}(k_B T)^2\frac{1}{\epsilon_F^2} + \cdots\right] = \frac{2}{3}\epsilon_F^{3/2}.$$

From this we get for the chemical potential the expression

$$\mu(T) = \epsilon_F\left[1 - \frac{\pi^2}{12}\left(\frac{k_B T}{\epsilon_F}\right)^2 + \cdots\right].$$

Employing again the Sommerfeld expansion we get

$$\begin{aligned} &\int_0^\infty d\epsilon\,\frac{\epsilon^{3/2}}{e^{\beta(\epsilon-\mu)}+1} \\ &\quad\approx \frac{2}{5}\mu^{5/2} + \frac{\pi^2}{4}(k_B T)^2\sqrt{\mu} + \cdots \\ &\quad= \frac{2}{5}\epsilon_F^{5/2}\left[1 + \frac{5}{12}\pi^2\left(\frac{k_B T}{\epsilon_F}\right)^2\right] + \cdots. \end{aligned}$$

Now the energy/particle is

$$\begin{aligned} \bar{\epsilon}(T) &= \frac{\int_0^\infty d\epsilon\,\frac{\epsilon^{3/2}}{e^{\beta(\epsilon-\mu)}+1}}{\int_0^\infty d\epsilon\,\frac{\epsilon^{1/2}}{e^{\beta(\epsilon-\mu)}+1}} \\ &= \frac{3}{5}\epsilon_F\left[1 + \frac{5}{12}\pi^2\left(\frac{k_B T}{\epsilon_F}\right)^2\right] \\ &= \frac{3}{5}\epsilon_F + \frac{\pi^2}{4}\frac{k_B^2 T^2}{\epsilon_F}. \end{aligned}$$

The heat capacity which can be written as

$$\begin{aligned} C_V &= \frac{\partial N\bar{\epsilon}}{\partial T} = N\frac{\pi^2}{2}\frac{k_B^2}{\epsilon_F}T \\ &= N k_B \frac{\pi^2}{2}\frac{T}{T_F}, \end{aligned}$$

is small when compared e.g. with the specific heat of the Maxwell-Boltzmann gas ($C_V = N k_B \frac{3}{2}$). This is understandable since the number of those particles that can be excited with the thermal energy $\sim k_B T$ is $\ll$ MB or BE gases due to the Pauli exclusion principle.

Pauli's paramagnetism

The magnetic moment of the electron is

$$\boldsymbol{\mu} = -\frac{e}{m}\,\boldsymbol{s}$$

or

$$\mu_z = -\mu_B \sigma_z,$$

where

$$\mu_B = \frac{e\hbar}{2m} = 5.66 \cdot 10^{-5} \frac{\mathrm{eV}}{\mathrm{T}}$$

and

$$\sigma_z = \frac{2}{\hbar}\, s_z = \pm 1.$$

In an external magnetic field the energy of an electron is

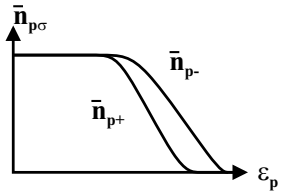
$$\epsilon_{\boldsymbol{p}\sigma_z} = \epsilon_{p\pm} = \frac{\boldsymbol{p}^2}{2m} - \mu_z B = \epsilon_{\boldsymbol{p}} + \mu_B B \sigma_z$$

when the kinetic energy is

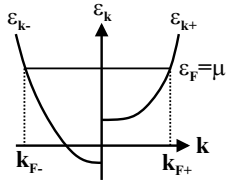
$$\epsilon_{\boldsymbol{p}} = \frac{\boldsymbol{p}^2}{2m}.$$

We still treat electrons as non interacting so the grand canonical partition function is as before, provided that we replace $\epsilon_{\boldsymbol{p}} \to \epsilon_{\boldsymbol{p}} + \mu_B B \sigma_z$.
The occupation numbers of the states are now

$$\bar{n}_{\boldsymbol{p}\sigma_z} = \bar{n}_{p\pm} = \frac{1}{e^{\beta(\epsilon_p + \mu_B B \sigma_z - \mu)} + 1}.$$



Since the metallic electron gas is strongly degenerated ($T \ll T_F$), we can restrict to the temperature $T = 0$.



The Fermi wave vectors can be determined from the conditions

$$\begin{aligned} \frac{\hbar^2 k_{F+}^2}{2m} + \mu_B B &= \mu \\ \frac{\hbar^2 k_{F-}^2}{2m} - \mu_B B &= \mu. \end{aligned}$$

Since the number density is

$$\rho = \frac{g_s}{6\pi^2}\, k_F^3,$$

the spin population densities are

$$\begin{aligned} \rho_+ &= \frac{k_{F+}^3}{6\pi^2} \\ \rho_- &= \frac{k_{F-}^3}{6\pi^2}. \end{aligned}$$

If the strength of the magnetic field is

$$B_0 = \frac{\epsilon_F}{\mu_B},$$

the magnetic energy is of the same order as the Fermi energy. For metals $\epsilon_F \approx 5\mathrm{eV}$, so $B_0 \approx 10^5 \mathrm{T}$. So the realistic magnetic fields are $\ll B_0$ and we can work at the small B limit. Let us denote

$$k_{F\pm} = k_F \pm \delta k_F,$$

so

$$\begin{aligned} \frac{\hbar^2 k_{F\pm}^2}{2m} \pm \mu_B B &= \frac{\hbar^2 k_F^2}{2m} \pm \frac{\hbar^2 k_F}{m}\,\delta k_F \pm \mu_B B \\ &= \mu = \frac{\hbar^2 k_F^2}{2m}. \end{aligned}$$

From this we get

$$\delta k_F = -\frac{m\mu_B}{\hbar^2 k_F}\, B$$

and

$$\begin{aligned} \rho_\pm &= \frac{k_F^3}{6\pi^2} \pm \frac{k_F^2}{2\pi^2}\,\delta k_F \\ &= \frac{k_F^3}{6\pi^2} \mp \frac{k_F m \mu_B}{2\pi^2 \hbar^2}\, B. \end{aligned}$$

The *relative polarization* is

$$\begin{aligned} r &\equiv \frac{\rho_+ - \rho_-}{\rho_+ + \rho_-} = -\frac{3m\mu_B}{\hbar^2 k_F^2}\, B \\ &= -\frac{3\mu_B}{2\epsilon_F}\, B. \end{aligned}$$

The magnetization per volume element is

$$M = \frac{N}{V}\,\langle \mu_z \rangle = -\rho \mu_B \langle \sigma_z \rangle = -\rho \mu_B r$$

or

$$M = \frac{3}{2}\,\rho \frac{\mu_B^2}{\epsilon_F}\, B.$$

The susceptivity is, according to its definition,

$$\chi = \frac{\partial M}{\partial H} = \mu_0 \frac{\partial M}{\partial B}.$$

Pauli's paramagnetic susceptivity is then

$$\chi = \frac{3}{2}\,\mu_0 \rho\, \frac{\mu_B^2}{\epsilon_F}$$

provided that $T \ll T_F$ and $\mu_B B \ll \epsilon_F$.
In aluminium the electron density is

$$\rho = 1.82 \cdot 10^{29} \mathrm{m}^{-3}$$

and the Fermi energy

$$\epsilon_F = \left(\frac{1.92}{2.07}\right)^2 13.6\mathrm{eV} = 11.7\mathrm{eV}.$$

The susceptivity

$$\begin{aligned}\chi &= \frac{3}{2}\cdot 4\pi\cdot 10^{-7}\cdot 1.82\cdot 10^{29}\cdot\frac{(5.66\cdot 10^{-5})^2}{11.7}\\ &\quad \frac{\mathrm{Vs}}{\mathrm{Am}}\,\frac{1}{\mathrm{m}^3}\,\frac{(\mathrm{eV})^2}{\mathrm{T}^2\mathrm{eV}}\\ &= 9.4\cdot 10^{13}\frac{\mathrm{eV\,Vs}}{\mathrm{Am}^4}\left(\frac{\mathrm{m}^2}{\mathrm{Vs}}\right)^2\\ &= 9.4\cdot 10^{13}\cdot 1.6\cdot 10^{-19}\\ &= 1.5\cdot 10^{-5}\end{aligned}$$

is now small since only the electrons very close to the Fermi surface can be polarized magnetically.

Two dimensional electron gas

The Hamiltonian for a free electron in the magnetic field

$$\boldsymbol{B} = \nabla\times\boldsymbol{A}$$

is given by

$$\mathcal{H}_0 = \frac{1}{2m^*}\left(-\mathrm{i}\hbar\nabla + \frac{e}{c}\boldsymbol{A}\right)^2.$$

Convenient unit of

- the energy for non-interacting electrons is $\hbar\omega_c$, where $\omega_c = eB/m^*c$ is the cyclotron frequency.
- the energy for interacting electrons is $e^2/\epsilon\ell_0$, where
- the length is $\ell_0 = (\hbar c/eB)^{\frac{1}{2}}$, the magnetic length.

Consider electrons

- confined to xy-plane.
- subjected to a perpendicular magnetic field $\boldsymbol{B}\|\hat{\boldsymbol{z}}$.

The eigenenergies are the discrete *Landau levels*

$$E_n = \left(n+\frac{1}{2}\right)\hbar\omega_c,\quad n = 0,1,2,\ldots.$$

Choosing the Landau gauge

$$\boldsymbol{A} = (0, Bx, 0)$$

the single particle Hamiltonian is

$$\mathcal{H}_0 = \frac{1}{2m^*}\left[p_x^2 + \left(p_y + \frac{eB}{c}x\right)^2\right].$$

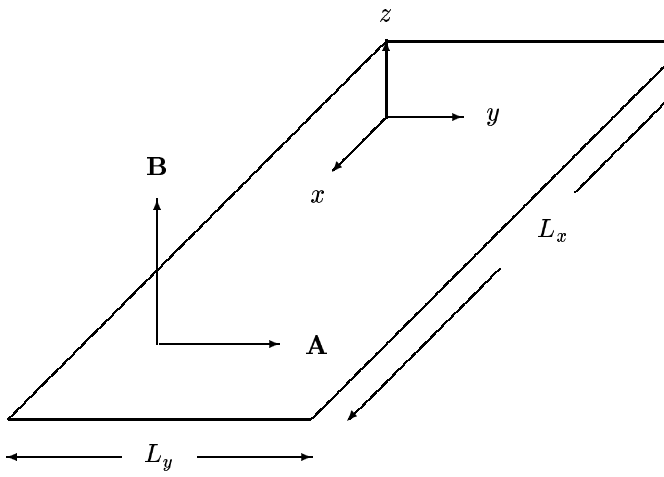
The eigenfunctions are

$$\phi_{nX} = e^{\mathrm{i}k_y y}e^{-(x-X)^2/2\ell_0^2}H_n\left(\frac{x-X}{\ell_0}\right),$$

where the center of the oscillatory motion is given by

$$X = -k_y\ell_0^2.$$

Confine the system in a rectangular cell



Using periodic boundary conditions we have

$$k_y = \frac{2\pi n_y}{L_y},\quad n_y = 0,\pm1,\pm2,\ldots$$

and

$$X = -\frac{2\pi n_y}{L_y}\ell_0^2,\quad 0\le X < L_x.$$

The number of allowed values of n_y, i.e. the degeneracy of each Landau level, is

$$N_s = \frac{L_xL_y}{2\pi\ell_0^2} = \frac{e}{hc}\boldsymbol{\Phi} = \frac{\boldsymbol{\Phi}}{\Phi_0},$$

where $\Phi_0 = hc/e$ is the flux quantum.
Thus, *on each Landau level there is exactly one state for each flux quantum and for each spin polarization.*
When N_e is the number of electrons in an area and N_s the number of flux quanta we define *the filling fraction* as

$$\nu = \frac{N_e}{N_s}\quad\left(= 4.136\,\frac{n_0}{10^{15}\mathrm{m}^{-2}}\,\frac{\mathrm{T}}{B}\right).$$

To treat the spin we note that

- there should be the Zeeman coupling term

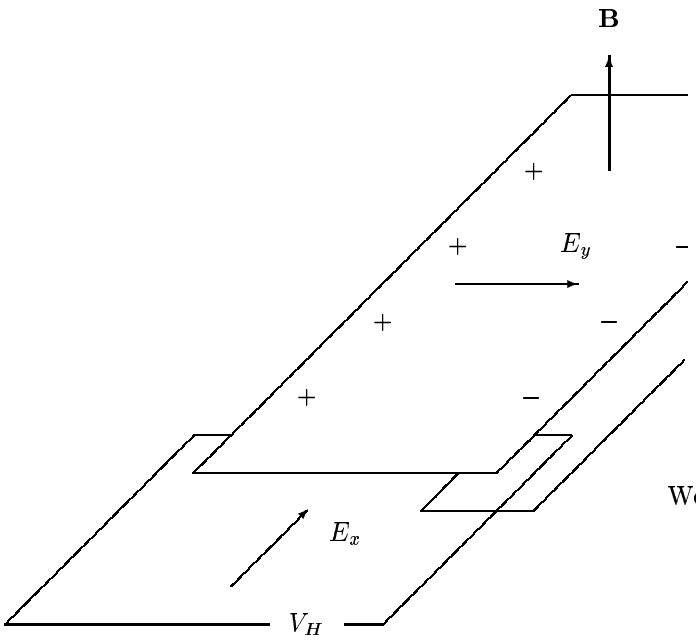
  $$\mathcal{H}_{\mathrm{Zeeman}} = \boldsymbol{\mu}\cdot\boldsymbol{B} = -g\mu_B B s_z$$

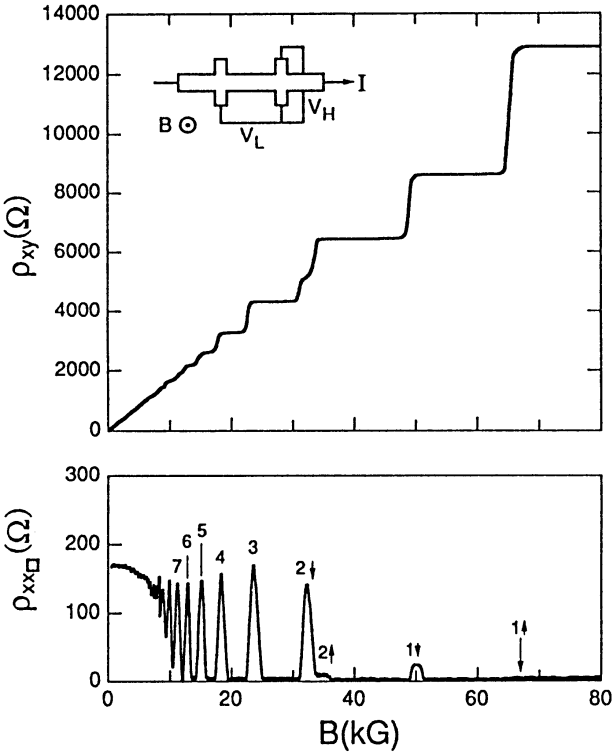
  in the Hamiltonian. Here g is the Lande factor and μ_B the Bohr magneton.
- in addition to the Zeeman term there are no spin dependent terms in the Hamiltonian, not even in the interacting many body system.
- the problem can be solved disregarding the spin. At later stages we can add the total Zeeman energy

  $$E_{\mathrm{Zeeman}} = g\mu_B B S_z.$$

Quantum Hall states

Consider an experiment like





We observe that

- the Hall resistivity develops plateaus with

$$\rho_{xy} = \frac{h}{ne^2}, \quad n = 1, 2, 3, \ldots.$$

  This quantization condition is obeyed with extreme accuracy. In fact, the current ISO standard for resistivity defines

$$\rho_{xy} = \frac{25812.807}{n}\,\Omega.$$

- at the same time the diagonal resistivity practically vanishes.

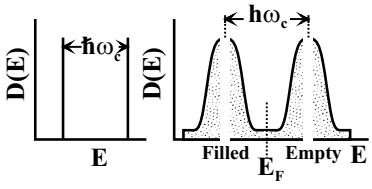
The conductivity σ and the resistivity ρ are defined by $\boldsymbol{j} = \sigma \boldsymbol{E}$, $\boldsymbol{E} = \rho \boldsymbol{j}$. Classically the diagonal and Hall conductivities are

$$\begin{aligned} \sigma_{xx} &= \frac{n_0 e^2 \tau}{m} \frac{1}{1 + (\omega_c \tau)^2} \\ \sigma_{xy} &= -\frac{n_0 e c}{B} + \frac{\sigma_{xx}}{\omega_c \tau}, \end{aligned}$$

where τ is the relaxation time. In particular $\rho_{xy} = -B/n_0 ec$.
Experimentally the resistivities behave like

For the moment we suppose that the electrons are polarized. If the current carrying electrons fill up exactly n Landau levels, it can be shown that $\rho_{xy} = h/ne^2$ and $\rho_{xx} = 0$.
The plateaus can be explained by noting that

- in an ideal pure 2DEG the density of states is a series of δ-peaks separated by $\hbar\omega_c$.
- In a real impure system the δ-peaks are spread and between the Landau levels there are localized states.



The Hall conductivity can be written in the form

$$\sigma_{xy} = -\frac{n_0 ec}{B} + \Delta\sigma_{xy},$$

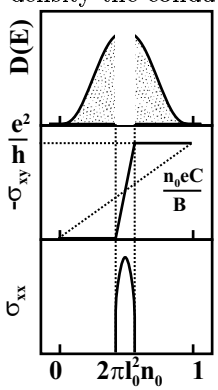
where, according to the Kubo formula, the contribution from a localized state $|\alpha\rangle$ to $\Delta\sigma_{xy}$ is

$$\Delta\sigma_{xy}^{\alpha} = \frac{f(E_\alpha)ec}{B}.$$

Here $f(E)$ is the Fermi distribution function.
When the number of electrons changes we observe (at $T = 0$) that

- as long as the Fermi level lies within the localized states, σ_{xy} remains constant.
- if all states below the Fermi level are localized, the terms in σ_{xy} cancel exactly and $\sigma_{xy} = 0$.
- for QHE to exist there must be extended states in Landau levels.

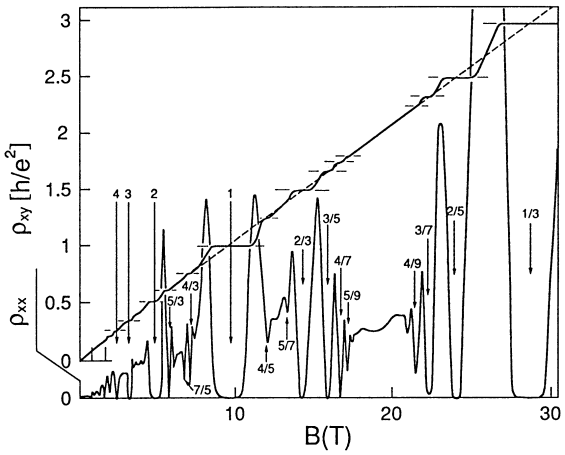
As a function of the density the conductivities behave like



Noting that

$$\nu = \frac{N_e}{N_s} = 2\pi\ell_0^2 n_0 \propto \frac{n_0}{B},$$

decreasing magnetic field corresponds to increasing filling factor, i.e. decreasing the magnetic field is equivalent to increasing the number of electrons.
Increasing the magnetic field (i.e. reducing the electron density) furthermore one finds resistivities to behave like



The plateaus in the Hall resistivity and the minima in the longitudinal resistivity correspond to filling fractions

$$\nu = \frac{p}{q},$$

where

- p and q are small integers ($\lesssim 11$).
- q is an *odd* integer.

This behaviour is called as the *Fractional Quantum Hall Effect* (FQHE) as opposed to the previous *Integer Quantum Hall Effect* (IQHE).
Regarding the IQHE we note that

- the plateaus correspond to full Landau levels,
- the Landau levels are energetically far from each other as compared to typical electron-electron interaction energies (at least when $\nu \lesssim 5$).
- the mutual electronic interactions play practically no role.

While this single particle picture is sufficient in the IQHE it cannot explain the FQHE where

- the Landau levels are only partially filled, so that
- there is room for the Coulomb intra level interaction.

It turns out that the correlations due to the electron interaction are essential in the FQHE.

Laughlin's theory
For a while we work in the symmetric gauge

$$\boldsymbol{B} = \frac{1}{2}(-y, x, 0)$$

and in the cylindrical coordinate system. The single particle Hamiltonian is now

$$\mathcal{H} = \frac{\boldsymbol{p}^2}{2m^*} + \frac{1}{2}\omega_c p_\phi + \frac{1}{8}m^*\omega_c^2 r^2.$$

The Schrödinger equation takes the form

$$-\frac{\hbar^2}{2m^*}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right)\frac{1}{r^2}\frac{\partial^2\psi}{\partial\phi^2}\right] - \frac{1}{2}i\hbar\omega_c\frac{\partial\psi}{\partial\phi} + \left(\frac{1}{8}m^*\omega_c^2 r^2 - E\right)\psi = 0.$$

Its solutions can be written as

$$\psi_{n,m}(r,\phi) = \left[\frac{n!}{2\pi\ell_0^2 2^m m!}\right]^{\frac{1}{2}} e^{-im\phi - r^2/4\ell_0^2} \times \left(\frac{r}{\ell_0}\right)^{|m|} L_n^{|m|}\left(\frac{r^2}{2\ell_0^2}\right).$$

The corresponding energies are

$$E_{n,m} = \frac{1}{2}(2n + |m| + 1 - m)\hbar\omega_c.$$

In particular, in the lowest Landau level ($n = 0, m \geq 0$), the wave functions are

$$\psi_m(z) = \left[\frac{1}{2\pi\ell_0^2 2^m m!}\right]^{\frac{1}{2}}\left(\frac{z}{\ell_0}\right)^m e^{-|z|^2/4\ell_0^2},$$

where we have written

$$z = re^{-i\phi} = x - iy.$$

It is easy to show, that the quantum number m can take the values

$$m = 0, 1, \ldots, N_s - 1,$$

where

$$N_s = \frac{A}{2\pi\ell_0^2}$$

is the degeneracy of a Landau level. In the lowest Landau level the wave functions are therefore of the form

$$1, z, z^2, \ldots, z^{N_s-1} \text{ times Gaussian.}$$

The great idea of Laughlin was to propose the Jastrow type function

$$\psi_m = \prod_{j<k}^{N_e} (z_j - z_k)^m \prod_{j=1}^{N_e} e^{-|z_j|^2/4\ell_0^2}$$

as the many body ground state wave function. To get the Fermi statistics m must be odd.
Laughlin's wave function has some remarkable properties:

- in the thermodynamical limit the parameter m is related to the filling fractio ν as

$$m = \frac{1}{\nu}.$$

- it can be mapped to a charge neutral two dimensional classical plasma, which makes it possible to use classical statistical mechanics to evaluate e.g. the energy.
- small systems ($\lesssim$12 particles) can be solved exactly. Comparisions with Laughlin's wave function show that it is practically the exact solution of the many body problem

Spin polarization

We consider the filling fraction $\nu = 1$, i.e. the lowest Landau level is fully occupied. We turn on the electron-electron interaction and note that

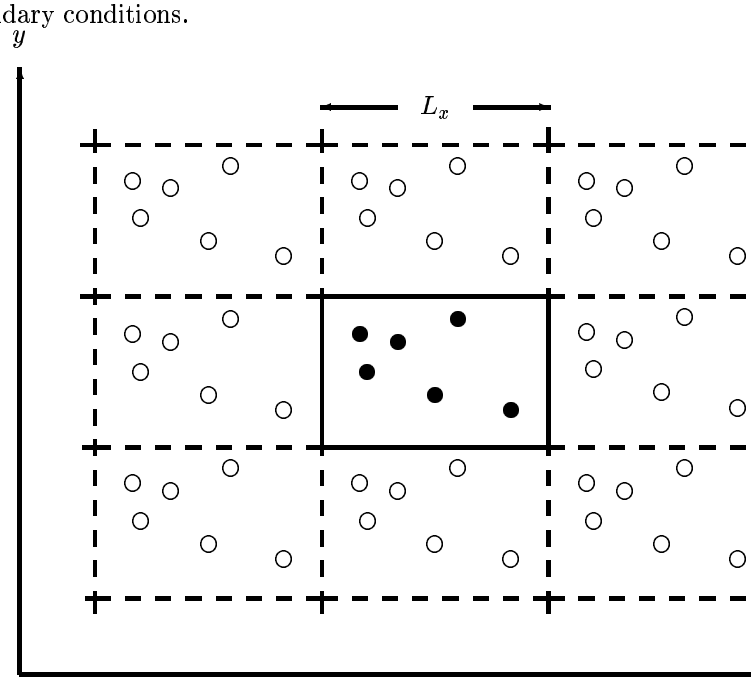
- typically the Landau level separation $\hbar\omega_c$ is (much) larger than the characteristic Coulomb interaction energy $e^2/\epsilon\ell_0$.
- if the electrons remain polarized the interaction cannot do much: all energetically favorable states are already occupied.

So, we let electrons to flip their spins. However,

- according to Hund's rule the repulsive interaction is the smaller the larger the total spin S.
- In the absence of the Zeeman coupling all possible S_z states are degenerate.
- the Zeeman coupling $g\mu_B BS_z$ tends to polarize the system, although the Lande g-factor is rather small (in GaAs $g \approx 0.5$).

We conclude that the ground state at $\nu = 1$ is polarized.

The diagonalization method
We will work in rectangular geometry with periodical boundary conditions.



Our Hamiltonian is

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{\text{e-e}} + \mathcal{H}_{\text{e-im}} + \mathcal{H}_{\text{e-b}} + \mathcal{H}_{\text{b-b}},$$

where

- we suppose a homogenous positive background,
- $\mathcal{H}_0$ is the single particle Hamiltonian,
- $\mathcal{H}_{\text{e-e}}$ is the Coulomb interaction between an electron and all other electrons and their images summed over all electrons,
- $\mathcal{H}_{\text{e-im}}$ is the interaction between an electron and its images,
- $\mathcal{H}_{\text{e-b}}$ is the electron-background interaction,
- $\mathcal{H}_{\text{b-b}}$ is the background-background interaction.

The Zeeman coupling is treated afterwards.
We

1. restrict to the lowest Landau level.
2. work in occupation representation. There

$$\begin{aligned}\mathcal{H} &= \sum_{j\sigma} w_j a^\dagger_{j\sigma} a_{j\sigma} \\ &\quad + \sum_{\substack{j_1\sigma_1 \quad j_2\sigma_2 \\ j_3\sigma_3 \quad j_4\sigma_4}} A_{j_1 j_2 j_3 j_4} a^\dagger_{j_1\sigma_1} a^\dagger_{j_2\sigma_2} a_{j_3\sigma_3} a_{j_4\sigma_4},\end{aligned}$$

where operators $a^\dagger_{j\sigma}$ ($a_{j\sigma}$) create (destroy) an electron with spin σ in a single particle state j.

3. fix N_s, the number of flux quanta (≈ 10). This is also the number of allowed single particle states.
4. fix N_e, the number of electrons. At full Landau level ($\nu = 1$) $N_e = N_s$.
5. fix the polarization S_z and the total momentum since they are preserved by Coulomb interaction.
6. form the basis by constructing all possible non-interacting states satisfying the above conditions.
7. represent the Hamiltonian as a matrix in the basis constructed above.
8. diagonalize the matrix. As a result we have the energy spectrum and corresponding eigenvectors.
9. for each eigenstate find its total spin S. Since $[\mathcal{H}, S] = 0$, we know that these eigenstates are eigenstates of spin, too.

$\langle S_z \rangle$

We now have the spectrum $E_0, E_1, E_2, \ldots$ for the interacting many particle system. To calculate the polarization we note that

- the energies E_i are associated with other quantum numbers like the total spin S_i and its z-component S_{zi}.
- since there are no spin dependent term in the Hamiltonian all states with quantum numbers $(E_i, S_i, S_{zi} = -S_i)$, $(E_i, S_i, S_{zi} = -S_i + 1)$, ..., $(E_i, S_i, S_{zi} = +S_i)$ are degenerate. So, the expectation value of S_z would be 0.
- the Zeeman interaction must be turned on. The energies will shift like

$$\epsilon_i = E_i - g\mu_B B S_{zi}.$$

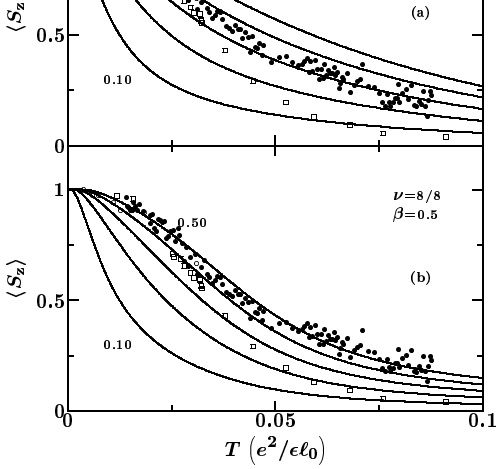
It turns out that, as expected, the total spin in the ground state is $S_0 = N_e/2$ (supposing N_e to be even). Due to the Zeeman coupling the ground state is polarized at $T = 0$. The spins of the excited states, however, have all the possible values $0, 1, \ldots, N_e/2$. So, we expect the polarization to decrease with increasing temperature. The dependence on temperature is evaluated in the canonical ensemble as

$$\langle S_z \rangle = \frac{1}{Z} \sum_i S_{zi} e^{-(E_i - g\mu_B B S_{zi})/k_B T},$$

where Z is the canonical partition sum

$$Z = \sum_i e^{-(E_i - g\mu_B B S_{zi})/k_B T}.$$

$\langle S_z \rangle$ at $\nu = 1$



Relativistic electron gas

The rest energy of an electron is

$$mc^2 = 0.511\text{keV}$$

and the relativistic total energy

$$\begin{aligned} \epsilon_p &= \sqrt{(mc^2)^2 + (cp)^2} \\ &= mc^2 + \frac{p^2}{2m} + \cdots. \end{aligned}$$

Denote by

$$k_c = \frac{mc}{\hbar} = 2.59 \cdot 10^{12}\text{m}^{-1}$$

the Compton wave vector of an electron and by

$$\lambda_c = \frac{2\pi}{k_c} = 2.43 \cdot 10^{-12}\text{m}$$

its Compton wave length.
Since $p = \hbar k$, we have

$$\epsilon_k = c\hbar\sqrt{k^2 + k_c^2}.$$

Periodic boundary conditions are the same as in the non relativistic case i.e.

$$\boldsymbol{k} = \frac{2\pi}{L}(n_x, n_y, n_z),$$

so we have

$$\rho = \frac{k_F^3}{3\pi^2}.$$

When $k_F = (3\pi^2\rho)^{1/3}$ is of the order k_c, the relativistic corrections must be taken into account. The corresponding density is

$$\begin{aligned} \rho_c &= \frac{k_c^3}{3\pi^2} = 5.87 \cdot 10^{35} \frac{1}{\text{m}^3} \\ &\approx 10^6 \times \text{density of metallic electron gas} \end{aligned}$$

We have an *ultrarelativistic electron gas* when $k_F \gg k_c$ or correspondingly $\rho \gg \rho_c$.
Let us consider *cold* relativistic material, i.e. let us suppose $T \ll T_F$.
The total energy is

$$E = N\bar{\epsilon} = N \frac{\int_0^{k_F} dk\, k^2 c\hbar\sqrt{k^2 + k_c^2}}{\int_0^{k_F} dk\, k^2},$$

where

$$\begin{aligned}\bar{\epsilon} &= mc^2 \frac{\int_0^{k_F/k_c} dx\, x^2 \sqrt{1+x^2}}{\int_0^{k_F/k_c} dx\, x^2} \\ &= mc^2 + mc^2 \frac{\int_0^{k_F/k_c} dx\, x^2 \left[\sqrt{1+x^2}-1\right]}{\int_0^{k_F/k_c} dx\, x^2}\end{aligned}$$

is the average electronic energy.
At the non relativistic limit we have

$$\begin{aligned}\bar{\epsilon} &\approx mc^2 \left[1 + \frac{\int_0^{k_F/k_c} dx\, x^2 [\frac{1}{2} x^2 + \cdots]}{\int_0^{k_F/k_c} dx\, x^2}\right] \\ &= mc^2 \left[1 + \frac{3}{10}\left(\frac{k_F}{k_c}\right)^2 + \cdots\right],\end{aligned}$$

from which our earlier results can be derived, provided that the rest energy of electrons is taken into account.
At the ultrarelativistic limit $k_F \gg k_c$ we get

$$\bar{\epsilon} \approx mc^2 \frac{\int_0^{k_F/k_c} dx\, x^3}{\int_0^{k_F/k_c} dx\, x^2} = \frac{3}{4} c\hbar k_F.$$

Thus the energy density is

$$\frac{E}{V} = \frac{3}{4}(3\pi^2)^{1/3} c\hbar \rho^{4/3}$$

and the pressure

$$p = -\left(\frac{\partial E}{\partial V}\right)_N$$

at the *ultrarelativistic limit*

$$p = \frac{1}{3}\frac{E}{V} = \frac{1}{4}(3\pi^2)^{1/3} c\hbar \rho^{4/3}.$$

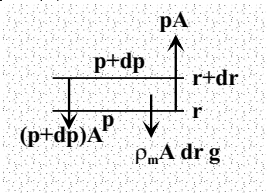
White dwarf
In a properly functioning star the energy released in nuclear reactions (mainly $2\mathrm{H} \to \mathrm{He}$) and the collapsing gravitational force are in balance. When the nuclear fuel is comsumed the start collapses. If the mass of the star is large eneough all material will become ionized. Depending on the mass of the star the final state can be for example

- *white dwarf*, if the pressure of the degenerated electronic plasma prohibits further compression.
- *neutron star* if the electronic pressure is not enough to compensate the gravitational force. The matter compresses further to neutrons and their degeneracy pressure prohibits further collapse.

Typical properties of a white dwarf:

- the diameter of the star $2R \approx 10^4\mathrm{km}$.
- the total number of nuclei $N_N \approx 10^{57}$.
- the mass $M \approx 10^{30}\mathrm{kg} \approx M_\odot$, where $M_\odot = 1.989 \cdot 10^{30}\mathrm{kg}$ is the mass of the sun.
- the mass density $\rho_m \approx 10^{10}\mathrm{kg\,m^{-3}}$ is about $10^6\times$ the density of the sun or of the earth.
- the number density of electrons $\rho \approx 10^{36}\mathrm{m^{-3}}$. Then $k_F \approx k_c$, so the electron gas is only moderately relativistic. In inner parts the gas can be much denser and thus ultrarelativistic.
- the pressure $p \approx 10^{22}\mathrm{Pa} \approx 10^{17}\mathrm{atm}$.
- the temperature in inner parts $T \approx 10^7\mathrm{K} \approx T_\odot$. Since the Fermi temperature is $T_F \approx 10^{10}\mathrm{K} \gg T$ we, however, have a cold electron gas.

Let $p(r)$ be the pressure at the distance r from the center of the star, $g(r)$ the corresponding gravitational acceleration and $\rho_m(r)$ the density.



The condition for the balance of hydrostatic mechanical forces is

$$\frac{dp}{dr} = -g(r)\rho_m(r).$$

Now

$$g(r) = \frac{GM(r)}{r^2},$$

where $M(r)$ is the mass inside of the radius r and

$$G = 6.673 \cdot 10^{-11} \frac{\mathrm{Nm^2}}{\mathrm{kg^2}}$$

is the gravitational constant. We get the pair of equations

$$\begin{aligned}\frac{dp(r)}{dr} &= -G\frac{M(r)\rho_m(r)}{r^2} \\ \frac{dM(r)}{dr} &= 4\pi r^2 \rho_m(r).\end{aligned}$$

Because in nuclei there are roughly as many neutrons as protons, and, on the other hand, there are as many protons as electrons, we have

$$\rho_m(r) \approx 2m_p\rho(r).$$

Here

$$m_p = 1.673 \cdot 10^{-27}\mathrm{kg}$$

is the proton mass and $\rho(r)$ the number density of the electrons.
As a good approximation the electron density of a star can be taken as a constant, ρ say. Then

$$M(r) = \frac{8}{3}\pi m_p \rho r^3$$

and thus the total mass

$$M = \frac{8}{3}\pi m_p \rho R^3,$$

when R is the radius of the star. The pressure must now satisfy the differential equation

$$\frac{dp}{dr} = -\frac{16}{3}\,\pi m_p^2 \rho^2 G r$$

with the boundary condition that the pressure vanishes at the surface, i.e.

$$p(R) = 0.$$

Integrating the differential equation we get for the pressure at the center

$$p = \frac{8\pi}{3}\,G m_p^2 \rho^2 R^2.$$

Since the electron gas is not quite ultrarelativistic we calculate more accurately than before. The average electronic energy is

$$\begin{aligned}
\bar{\epsilon} &= mc^2\,\frac{\int_0^{k_F/k_c} dx\, x^2\sqrt{x^2+1}}{\int_0^{k_F/k_c} dx\, x^2} \\
&= mc^2\,\frac{\int_0^{k_F/k_c} dx\, x^3\left[1+\frac{1}{2}\,\frac{1}{x^2}+\cdots\right]}{\int_0^{k_F/k_c} dx\, x^2} \\
&= mc^2\left[\frac{3}{4}\,\frac{k_F}{k_c}+\frac{3}{4}\,\frac{k_c}{k_F}+\cdots\right].
\end{aligned}$$

From this we can get for the pressure

$$\begin{aligned}
p &= \frac{\hbar c}{12\pi^2}\,(k_F^4 - k_c^2 k_F^2 + \cdots) \\
&= \frac{1}{4}\,(3\pi^2)^{1/3}\hbar c \rho^{4/3}\left[1 - \frac{m^2c^2}{\hbar^2(3\pi^2\rho)^{2/3}} + \cdots\right].
\end{aligned}$$

This the equation of state of the relativistic electron gas. We require that the pressures obtained from the equation of state and from the hydrodynamic balance conditions are equal in the center, i.e.

$$\frac{8\pi}{3}\,G m_p \rho^2 R^2 = \frac{1}{4}\,(3\pi^2)^{1/3}\hbar c \rho^{4/3}\left[1 - \frac{m^2c^2}{\hbar^2(3\pi^2\rho)^{2/3}} + \cdots\right].$$

When we substitute the electron density (as a function of the mass and radius)

$$\rho = \frac{3M}{8\pi m_p R^3}$$

we get the condition

$$\left(\frac{M}{M_c}\right)^{2/3} = 1 - \left(\frac{R}{R_c}\right)^2 \left(\frac{M_c}{M}\right)^{2/3},$$

where

$$\begin{aligned}
M_c &= m_p \left(\frac{9\pi}{512}\right)^{1/2}\left(\frac{\hbar c}{G m_p^2}\right)^{3/2} \approx 0.52\cdot 10^{57} m_p \\
R_c &= \frac{\hbar}{mc}\left(\frac{9\pi}{8}\right)^{1/3}\left(\frac{M_c}{m_p}\right)^{1/3} \approx 4700\text{km}.
\end{aligned}$$

For the radius of the star we get

$$R = R_c\left(\frac{M}{M_c}\right)^{1/3}\left[1 - \left(\frac{M}{M_c}\right)^{1/3}\right].$$

We see that the white dwarf has the maximum mass $M = M_c$. A more careful calculation shows that the mass of a white dwarf cannot exceed *Chandrasekhar's limit*, about $1.4 M_\odot$, without collapsing to a neutron star or a black hole.

Other Fermionic systems

Nuclear matter
The mass density of heavy nuclei is

$$\rho_m \approx 2.8\cdot 10^{17}\text{kg m}^{-3}.$$

When we assume that the proton and neutron densities are equal the Fermi wave vectors of both gases are

$$k_F \approx 1.36\cdot 10^{15}\text{m}^{-1}$$

and the Fermi energies

$$\epsilon_F \approx 38\text{MeV}.$$

Since $m_n c^2 = 938\text{MeV}$, the nuclear matter is non relativistic. The attractive nucleon interactions cancel the pressure due to the kinetic energy.

Neutron star
When the mass of a star exceeds the Chandrasekar limit the Fermi pressure of the electrons is not enough to cancel the gravitational force. The star continues its collapse. The star forms a giant nucleus where most electrons and protons have transformed via the reaction

$$\text{p} + \text{e}^- \to \text{n} + \nu_\text{e}$$

to neutrons. The radius of the star is

$$R \approx 10\text{km},$$

the nucleon count

$$N_N \approx 10^{57}$$

and the mass density

$$\rho_m \approx 10^{18}\text{kg m}^{-3}.$$

The pressure acting against the graviation is mostly due to the pressure of the Fermi gas and to the strong, at short distances very repulsive nuclear forces.

Quark matter
When nuclear matter is compressed 2–10 times denser that in atomic nuclei. the nucleons start to "overlap" and their constituent quarks form a quark plasma.

Liquid ^{3}He
The nucleus is p+p+n and the nuclear spin $\frac{1}{2}$.

At low temperatures the nuclear spin determines the statistics, i.e. ^{3}He atoms are Fermions.
The Fermi temperature corresponding to the normal density is

$$T_F = \frac{\epsilon_F}{k_B} \approx 5\mathrm{K}.$$

Since the mutual interactions between ^{3}He atoms are considerable the ^{3}He matter forms an interacting *Fermi liquid*. The ^{3}He liquid has two super phases (A and B). These are in balance with the normal phase at the critical point

$$T_c \approx 2.7\mathrm{mK} < \frac{T_F}{1000}.$$