Yleinen suhteellisuusteoria, General relativity 763695S
Tentti, Examination 2.4.2004

1. (a) Define the manifold.
   (b) Define the covariant vector.
   (c) Show that the property $\tau^{ab} = \tau^{ba}$ of a tensor is independent of the coordinate system

2. Derive an expression for the absolute derivative of a covariant vector μ_a starting from the expression of absolute derivative of a contravariant vector and the fact that $\lambda^a \mu_a$ is a scalar.

3. We studied the equation

$$(\rho u^\mu)_{,\mu} + (p/c^2) u^\mu{}_{,\mu} = 0 \tag{1}$$

   where u^μ is the four velocity. Derive the equation to which this reduces in case of ordinary nonrelativistic materia.

4. A uniform gravitational field $V = gz$ corresponds to metric

$$c^2 d\tau^2 = c^2(1 + \frac{2gz}{c^2})dt^2 - dx^2 - dy^2 - dz^2. \tag{2}$$

   Calculate the coefficients $\Gamma^\mu_{\nu\sigma}$ using variables $x^0 = t$, $x^1 = x$, $x^2 = y$ and $x^3 = z$. Show that in the limit $c \to \infty$ the geodetic lines satisfy Newton's equation of motion

$$m\frac{d^2\mathbf{r}}{dt^2} = -mg\mathbf{e}_3 \tag{3}$$

5. The Friedmann equation

$$\dot{R}^2 + k = \frac{8\pi G}{3}\rho R^2 \tag{4}$$

   was derived in the course (at least partly). Explain the quantities appearing in this equation. Explain what equations are needed to derive this expression. (Do not make the calculations, they are too long.)

Fill in the course evaluation form!

Collection of formulas

$$\Gamma^a_{bc} = \tfrac{1}{2} g^{ad} \left(\partial_b g_{dc} + \partial_c g_{bd} - \partial_d g_{bc}\right) \tag{5}$$

$$\frac{d^2 x^a}{du^2} + \Gamma^a_{bc} \frac{dx^b}{du} \frac{dx^c}{du} = 0 \tag{6}$$

$$\frac{d}{du}\left(\frac{\partial L}{\partial \dot{x}^c}\right) - \frac{\partial L}{\partial x^c} = 0, \qquad L\left(\dot{x}^c, x^c\right) \equiv \tfrac{1}{2} g_{ab}\left(x^c\right) \dot{x}^a \dot{x}^b \tag{7}$$

$$\frac{D\lambda^a}{du} = \frac{d\lambda^\alpha}{du} + \Gamma^a_{bc}\lambda^b \frac{dx^c}{du} \tag{8}$$

$$\tau^a_{b;c} = \partial_c \tau^a_b + \Gamma^a_{dc}\tau^d_b - \Gamma^d_{bc}\tau^a_d \tag{9}$$

$$\lambda_{a;bc} - \lambda_{a;cb} = R^d{}_{abc}\lambda_d \tag{10}$$

$$R^d{}_{abc} = \partial_b \Gamma^d_{ac} - \partial_c \Gamma^d_{ab} + \Gamma^e_{ac}\Gamma^d_{eb} - \Gamma^e_{ab}\Gamma^d_{ec} \tag{11}$$

$$R_{abcd} = -R_{bacd} = -R_{abdc} = R_{cdab} \tag{12}$$

$$R^a{}_{bcd} + R^a{}_{cdb} + R^a{}_{dbc} = 0 \tag{13}$$

$$R_{ab} = R^c{}_{abc} \qquad R = g^{ab}R_{ab} \qquad G_{ab} = R_{ab} - \tfrac{1}{2}Rg_{ab} \tag{14}$$

$$R^{\mu\nu} - \tfrac{1}{2}Rg^{\mu\nu} = \kappa T^{\mu\nu}, \quad \kappa = -\frac{8\pi G}{c^4} \tag{15}$$

$$T^{\mu\nu} = (\rho + \frac{p}{c^2})u^\mu u^\nu - pg^{\mu\nu} \tag{16}$$

$$c^2 d\tau^2 = \left(1 - \frac{2m}{r}\right)c^2 dt^2 - \frac{dr^2}{1 - \frac{2m}{r}} - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2, \quad m = \frac{GM}{c^2} \tag{17}$$

$$d\tau^2 = dt^2 - R(t)^2\left(\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right), \quad c = 1 \tag{18}$$