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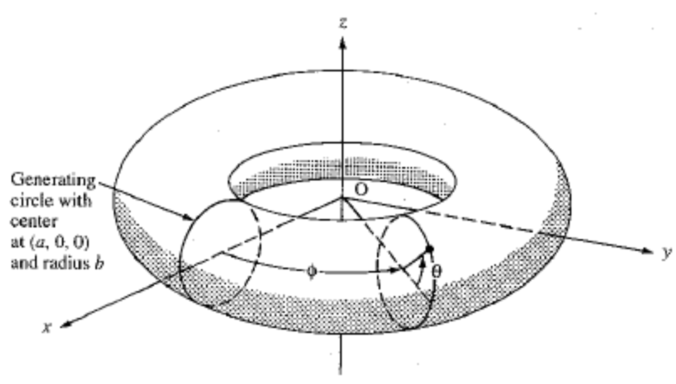
Tentti, Examination 11.12.2009

1. (a) How is a contravariant vector defined on a manifold?
   (b) The curl of a covariant vector field λ_a is defined as the skew symmetric tensor field $\lambda_{a;b} - \lambda_{b;a}$. Show that
   $$\lambda_{a;b} - \lambda_{b;a} = \partial_b \lambda_a - \partial_a \lambda_b \tag{1}$$
   (c) What is meant by local cartesian coordinates on a manifold?

2. We define in $x - y$ plane the coordinates (u, v) by equations
   $$x = \sqrt{uv}, \qquad y = \sqrt{\frac{u}{v}}. \tag{2}$$
   For coordinates (u, v), calculate the natural basis vectors and the corresponding dual basis vectors expressed in the basis $(\boldsymbol{i}, \boldsymbol{j})$ of cartesian coordinates. Which of the basis vectors are everywhere orthogonal to each other?

3. For a torus (see figure) one gets the line element
   $$ds^2 = b^2 d\theta^2 + (a + b\cos\theta)^2 d\phi^2, \tag{3}$$
   where $b < a$. Derive the equations for geodesic lines using Lagrange equations and calculate the connection coefficients. Study whether the lines $\theta =$ constant are geodesic.



4. Starting from the line element
   $$ds^2 = A(r)dt^2 + B(r)dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2, \tag{4}$$
   what should be done in order to derive the Schwarzschild metric around a massive object. Do not attempt to do the calculations, tell only what equations you have to use.

5. We study the Schwarzschild metric around a massive body: compare the distances in different directions and the time in stationary clocks at different distances compared to the Schwarzschild coordinates.

Fill in the course evaluation form!

Collection of formulas (NOTE THE ADDITIONS)

$$\boldsymbol{e}_i = \frac{\partial \boldsymbol{r}}{\partial u^i}, \qquad \boldsymbol{e}^i = \boldsymbol{\nabla} u^i, \qquad g_{ij} = \boldsymbol{e}_i \cdot \boldsymbol{e}_j \tag{5}$$

$$\Gamma^a_{bc} = \tfrac{1}{2} g^{ad} \left(\partial_b g_{dc} + \partial_c g_{bd} - \partial_d g_{bc} \right) \tag{6}$$

$$\frac{d^2 x^a}{du^2} + \Gamma^a_{bc} \frac{dx^b}{du} \frac{dx^c}{du} = 0 \tag{7}$$

$$\frac{d}{du}\left(\frac{\partial L}{\partial \dot{x}^c} \right) - \frac{\partial L}{\partial x^c} = 0, \qquad L\left(\dot{x}^c, x^c\right) \equiv \tfrac{1}{2} g_{ab}\left(x^c\right) \dot{x}^a \dot{x}^b \tag{8}$$

$$\frac{D\lambda^a}{du} = \frac{d\lambda^\alpha}{du} + \Gamma^a_{bc} \lambda^b \frac{dx^c}{du} \tag{9}$$

$$\tau^a_{b;c} = \partial_c \tau^a_b + \Gamma^a_{dc} \tau^d_b - \Gamma^d_{bc} \tau^a_d \tag{10}$$

$$\lambda_{a;bc} - \lambda_{a;cb} = R^d{}_{abc} \lambda_d \tag{11}$$

$$R^d{}_{abc} = \partial_b \Gamma^d_{ac} - \partial_c \Gamma^d_{ab} + \Gamma^e_{ac} \Gamma^d_{eb} - \Gamma^e_{ab} \Gamma^d_{ec} \tag{12}$$

$$R_{abcd} = -R_{bacd} = -R_{abdc} = R_{cdab} \tag{13}$$

$$R^a{}_{bcd} + R^a{}_{cdb} + R^a{}_{dbc} = 0 \tag{14}$$

$$R_{ab} = R^c{}_{abc} \qquad R = g^{ab} R_{ab} \qquad G_{ab} = R_{ab} - \tfrac{1}{2} R g_{ab} \tag{15}$$

$$R^{\mu\nu} - \tfrac{1}{2} R g^{\mu\nu} = \kappa T^{\mu\nu}, \qquad \kappa = -\frac{8\pi G}{c^4} \tag{16}$$

$$T^{\mu\nu} = (\rho + \frac{p}{c^2}) u^\mu u^\nu - p g^{\mu\nu} \tag{17}$$

$$c^2 d\tau^2 = \left(1 - \frac{2m}{r}\right) c^2 dt^2 - \frac{dr^2}{1 - \frac{2m}{r}} - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2, \quad m = \frac{GM}{c^2} \tag{18}$$

$$d\tau^2 = dt^2 - R(t)^2 \left(\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \right), \qquad c = 1 \tag{19}$$